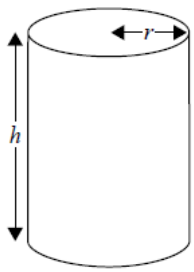
Optimization *[153 marks]*

A closed cylindrical can with radius r centimetres and height h centimetres has a volume of 20 π cm^3.

diagram not to scale



1a. Express *h* in terms of *r.* *[2 marks]*

Markscheme

correct equation for volume ***(A1)***
eg $\pi r^2 h = 20\pi$

$h = \frac{20}{r^2}$ ***A1 N2***

[2 marks]

The material for the base and top of the can costs 10 cents per cm ^{2} and the material for the curved side costs 8 cents per cm^2. The total cost of the material, in cents, is *C*.

1b. Show that $C = 20\pi r^2 + \frac{320\pi}{r}$. *[4 marks]*

Markscheme

attempt to find formula for cost of parts ***(M1)***
eg 10 × two circles, 8 × curved side

correct expression for cost of two circles in terms of *r* (seen anywhere) ***A1***
eg $2\pi r^2 \times 10$

correct expression for cost of curved side (seen anywhere) ***(A1)***
eg $2\pi r \times h \times 8$

correct expression for cost of curved side in terms of *r* ***A1***
eg $8 \times 2\pi r \times \frac{20}{r^2}, \ \frac{320\pi}{r^2}$

$C = 20\pi r^2 + \frac{320\pi}{r}$ ***AG N0***

[4 marks]

1c. Given that there is a minimum value for *C*, find this minimum value in terms of π. *[9 marks]*

Markscheme

recognize $C' = 0$ at minimum ***(R1)***
eg $C' = 0, \ \frac{\mathrm{d}C}{\mathrm{d}r} = 0$

correct differentiation (may be seen in equation)

$C' = 40\pi r - \frac{320\pi}{r^2}$ ***A1A1***

correct equation ***A1***
eg $40\pi r - \frac{320\pi}{r^2} = 0, \ 40\pi r\frac{320\pi}{r^2}$

correct working ***(A1)***
eg $40r^3 = 320, \ r^3 = 8$

r = 2 (m) ***A1***

attempt to substitute **their** value of *r* into *C*
eg $20\pi \times 4 + 320 \times \frac{\pi}{2}$ ***(M1)***

correct working
eg $80\pi + 160\pi$ ***(A1)***

240π (cents) ***A1 N3***

Note: Do not accept 753.6, 753.98 or 754, even if 240 π is seen.

[9 marks]

2. Consider $f(x) = \log k(6x - 3x^2)$, for $0 < x < 2$, where $k > 0$. *[7 marks]*

The equation $f(x) = 2$ has exactly one solution. Find the value of k.

Markscheme

METHOD 1 – using discriminant

correct equation without logs ***(A1)***

eg $6x - 3x^2 = k^2$

valid approach ***(M1)***

eg $-3x^2 + 6x - k^2 = 0,\ 3x^2 - 6x + k^2 = 0$

recognizing discriminant must be zero (seen anywhere) ***M1***

eg $\Delta = 0$

correct discriminant ***(A1)***

eg $6^2 - 4(-3)(-k^2),\ 36 - 12k^2 = 0$

correct working ***(A1)***

eg $12k^2 = 36,\ k^2 = 3$

$k = \sqrt{3}$ ***A2 N2***

METHOD 2 – completing the square

correct equation without logs ***(A1)***

eg $6x - 3x^2 = k^2$

valid approach to complete the square ***(M1)***

eg $3(x^2 - 2x + 1) = -k^2 + 3,\ x^2 - 2x + 1 - 1 + \frac{k^2}{3} = 0$

correct working ***(A1)***

eg $3(x - 1)^2 = -k^2 + 3,\ (x - 1)^2 - 1 + \frac{k^2}{3} = 0$

recognizing conditions for one solution ***M1***

eg $(x - 1)^2 = 0,\ -1 + \frac{k^2}{3} = 0$

correct working ***(A1)***

eg $\frac{k^2}{3} = 1,\ k^2 = 3$

$k = \sqrt{3}$ ***A2 N2***

[7 marks]

Let $f(x) = \ln x$ and $g(x) = 3 + \ln\left(\frac{x}{2}\right)$, for $x > 0$.

The graph of g can be obtained from the graph of f by two transformations:

$$\text{a horizontal stretch of scale factor } q \text{ followed by}$$
$$\text{a translation of } \begin{pmatrix} h \\ k \end{pmatrix}.$$

3a. Write down the value of q; *[1 mark]*

Markscheme

$q = 2$ ***A1*** ***N1***

Note: Accept $q = 1$, $h = 0$, and $k = 3 - \ln(2)$, 2.31 as candidate may have rewritten $g(x)$ as equal to $3 + \ln(x) - \ln(2)$.

[1 mark]

3b. Write down the value of h; *[1 mark]*

Markscheme

$h = 0$ ***A1*** ***N1***

Note: Accept $q = 1$, $h = 0$, and $k = 3 - \ln(2)$, 2.31 as candidate may have rewritten $g(x)$ as equal to $3 + \ln(x) - \ln(2)$.

[1 mark]

3c. Write down the value of k. *[1 mark]*

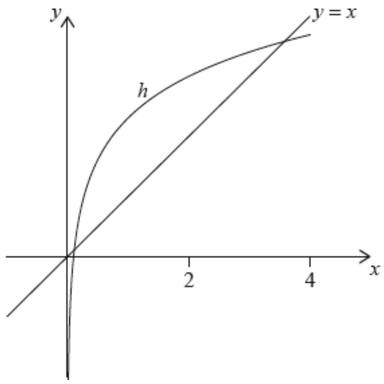
Markscheme

$k = 3$ ***A1*** ***N1***

Note: Accept $q = 1$, $h = 0$, and $k = 3 - \ln(2)$, 2.31 as candidate may have rewritten $g(x)$ as equal to $3 + \ln(x) - \ln(2)$.

[1 mark]

Let $h(x) = g(x) \times \cos(0.1x)$, for $0 < x < 4$. The following diagram shows the graph of h and the line $y = x$.



The graph of h intersects the graph of h^{-1} at two points. These points have x coordinates 0.111 and 3.31 correct to three significant figures.

3d. Find $\int_{0.111}^{3.31} (h(x) - x)\,\mathrm{d}x$. *[2 marks]*

Markscheme

2.72409

2.72 ***A2*** ***N2***

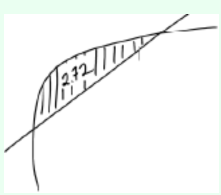
[2 marks]

3e. Hence, find the area of the region enclosed by the graphs of h and h^{-1}. *[3 marks]*

Markscheme

recognizing area between $y = x$ and h equals 2.72 ***(M1)***

eg



recognizing graphs of h and h^{-1} are reflections of each other in $y = x$ ***(M1)***

eg area between $y = x$ and h equals between $y = x$ and h^{-1}

$2 \times 2.72 \int_{0.111}^{3.31} \left(x - h^{-1}(x)\right) \mathrm{d}x = 2.72$

5.44819

5.45 ***A1*** ***N3***

[??? marks]

3f. Let d be the vertical distance from a point on the graph of h to the line $y = x$. There is a point $\mathrm{P}(a,\, b)$ on the graph of h where d is a maximum. *[7 marks]*

Find the coordinates of P, where $0.111 < a < 3.31$.

Markscheme

valid attempt to find d *(M1)*

eg difference in y-coordinates, $d = h(x) - x$

correct expression for d *(A1)*

eg $\left(\ln \frac{1}{2}x + 3\right)(\cos 0.1x) - x$

valid approach to find when d is a maximum *(M1)*

eg max on sketch of d, attempt to solve $d' = 0$

0.973679

$x = 0.974$ ***A2*** ***N4***

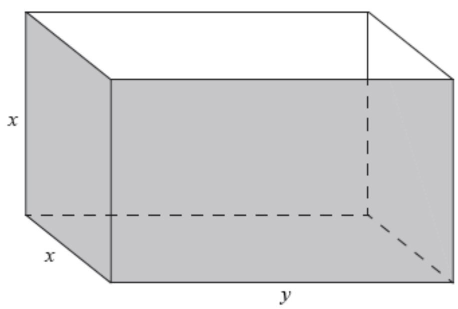
substituting **their** x value into $h(x)$ *(M1)*

2.26938

$y = 2.27$ ***A1*** ***N2***

[7 marks]

Fred makes an open metal container in the shape of a cuboid, as shown in the following diagram.



The container has height x m, width x m and length y m. The volume is $36\ \text{m}^3$.

Let $A(x)$ be the outside surface area of the container.

4a. Show that $A(x) = \frac{108}{x} + 2x^2$. *[4 marks]*

Markscheme

correct substitution into the formula for volume ***A1***

eg $36 = y \times x \times x$

valid approach to eliminate y (may be seen in formula/substitution) ***M1***

eg $y = \frac{36}{x^2},\ xy = \frac{36}{x}$

correct expression for surface area ***A1***

eg $xy + xy + xy + x^2 + x^2,\ \text{area} = 3xy + 2x^2$

correct expression in terms of x only ***A1***

eg $3x\left(\frac{36}{x^2}\right) + 2x^2,\ x^2 + x^2 + \frac{36}{x} + \frac{36}{x} + \frac{36}{x},\ 2x^2 + 3\left(\frac{36}{x}\right)$

$A(x) = \frac{108}{x} + 2x^2$ ***AG*** ***N0***

[4 marks]

4b. Find $A'(x)$. *[2 marks]*

Markscheme

$A'(x) = -\frac{108}{x^2} + 4x,\ 4x - 108x^{-2}$ ***A1A1*** ***N2***

Note: Award ***A1*** for each term.

[2 marks]

4c. Given that the outside surface area is a minimum, find the height of the container. *[5 marks]*

Markscheme

recognizing that minimum is when $A'(x) = 0$ ***(M1)***

correct equation ***(A1)***

eg $-\frac{108}{x^2} + 4x = 0,\ 4x = \frac{108}{x^2}$

correct simplification ***(A1)***

eg $-108 + 4x^3 = 0,\ 4x^3 = 108$

correct working ***(A1)***

eg $x^3 = 27$

$\text{height} = 3\ (\text{m})\ (\text{accept } x = 3)$ ***A1*** ***N2***

[5 marks]

4d. Fred paints the outside of the container. A tin of paint covers a surface area of $10\ \mathrm{m}^2$ *[5 marks]* and costs $20. Find the total cost of the tins needed to paint the container.

Markscheme

attempt to find area using **their** height ***(M1)***

eg $\frac{108}{3}+2(3)^2,\ 9+9+12+12+12$

minimum surface area $=54\ \mathrm{m}^2$ (may be seen in part (c)) ***A1***

attempt to find the number of tins ***(M1)***

eg $\frac{54}{10},\ 5.4$

6 (tins) ***(A1)***

$120 ***A1*** ***N3***

[5 marks]

Let L_x be a family of lines with equation given by $r=\begin{pmatrix} x \\ \frac{2}{x} \end{pmatrix}+t\begin{pmatrix} x^2 \\ -2 \end{pmatrix}$, where $x>0$.

5a. Write down the equation of L_1. *[2 marks]*

Markscheme

attempt to substitute $x=1$ ***(M1)***

eg $\boldsymbol{r}=\begin{pmatrix} 1 \\ \frac{2}{1} \end{pmatrix}+t\begin{pmatrix} 1^2 \\ -2 \end{pmatrix},\ L_1=\begin{pmatrix} 1 \\ 2 \end{pmatrix}+t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

correct equation (vector or Cartesian, but do not accept " L_1")

eg $\boldsymbol{r}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}+t\begin{pmatrix} 1 \\ -2 \end{pmatrix},\ y=-2x+4$ (must be an equation) ***A1*** ***N2***

[2 marks]

5b. A line L_a crosses the y-axis at a point P. *[6 marks]*

Show that P has coordinates $\left(0,\ \frac{4}{a}\right)$.

Markscheme

appropriate approach ***(M1)***

eg $\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} a \\ \frac{2}{a} \end{pmatrix} + t\begin{pmatrix} a^2 \\ -2 \end{pmatrix}$

correct equation for x-coordinate ***A1***

eg $0 = a + ta^2$

$t = \frac{-1}{a}$ ***A1***

substituting **their** parameter to find y ***(M1)***

eg $y = \frac{2}{a} - 2\left(\frac{-1}{a}\right)$, $\begin{pmatrix} a \\ \frac{2}{a} \end{pmatrix} - \frac{1}{a}\begin{pmatrix} a^2 \\ -2 \end{pmatrix}$

correct working ***A1***

eg $y = \frac{2}{a} + \frac{2}{a}$, $\begin{pmatrix} a \\ \frac{2}{a} \end{pmatrix} - \begin{pmatrix} a \\ -\frac{2}{a} \end{pmatrix}$

finding correct expression for y ***A1***

eg $y = \frac{4}{a}$, $\begin{pmatrix} 0 \\ \frac{4}{a} \end{pmatrix}$ $\mathrm{P}\left(0,\ \frac{4}{a}\right)$ ***AG*** ***N0***

[6 marks]

5c. The line L_a crosses the x-axis at $\mathrm{Q}(2a,\ 0)$. Let $d = \mathrm{PQ}^2$. *[2 marks]*

Show that $d = 4a^2 + \frac{16}{a^2}$.

Markscheme

valid approach ***M1***

eg distance formula, Pythagorean Theorem, $\overrightarrow{\mathrm{PQ}} = \begin{pmatrix} 2a \\ -\frac{4}{a} \end{pmatrix}$

correct simplification ***A1***

eg $(2a)^2 + \left(\frac{4}{a}\right)^2$

$d = 4a^2 + \frac{16}{a^2}$ ***AG*** ***N0***

[2 marks]

5d. There is a minimum value for d. Find the value of a that gives this minimum value. *[7 marks]*

Markscheme

recognizing need to find derivative *(M1)*

eg $d', d'(a)$

correct derivative ***A2***

eg $8a - \frac{32}{a^3}, 8x - \frac{32}{x^3}$

setting **their** derivative equal to 0 *(M1)*

eg $8a - \frac{32}{a^3} = 0$

correct working *(A1)*

eg $8a = \frac{32}{a^3}, 8a^4 - 32 = 0$

working towards solution *(A1)*

eg $a^4 = 4, a^2 = 2, a = \pm\sqrt{2}$

$a = \sqrt[4]{4} \quad (a = \sqrt{2}) \quad (\text{do not accept } \pm\sqrt{2})$ ***A1*** ***N3***

[7 marks]

Total [17 marks]

Let
$f(x) = \frac{3x}{x-q}$, where
$x \neq q$.

6a. Write down the equations of the vertical and horizontal asymptotes of the graph of f. *[2 marks]*

Markscheme

$x = q, y = 3$ (must be equations) ***A1A1*** ***N2***

[2 marks]

6b. The vertical and horizontal asymptotes to the graph of f intersect at the point $\mathrm{Q}(1, 3)$. *[2 marks]*

Find the value of q.

Markscheme

recognizing connection between point of intersection and asymptote *(R1)*

eg $x = 1$

$q = 1$ ***A1*** ***N2***

[2 marks]

6c. The vertical and horizontal asymptotes to the graph of f intersect at the point $\mathrm{Q}(1,3)$. *[4 marks]*

The point $\mathrm{P}(x,\ y)$ lies on the graph of f. Show that $\mathrm{PQ} = \sqrt{(x-1)^2 + \left(\frac{3}{x-1}\right)^2}$.

Markscheme

correct substitution into distance formula ***A1***

eg $\sqrt{(x-1)^2 + (y-3)^2}$

attempt to substitute $y = \frac{3x}{x-1}$ ***(M1)***

eg $\sqrt{(x-1)^2 + \left(\frac{3x}{x-1} - 3\right)^2}$

correct simplification of $\left(\frac{3x}{x-1} - 3\right)$ ***(A1)***

eg $\frac{3x - 3x(x-1)}{x-1}$

correct expression clearly leading to the required answer ***A1***

eg $\frac{3x-3x+3}{x-1}$, $\sqrt{(x-1)^2 + \left(\frac{3x-3x+3}{x-1}\right)^2}$

$\mathrm{PQ} = \sqrt{(x-1)^2 + \left(\frac{3}{x-1}\right)^2}$ ***AG*** ***N0***

[4 marks]

6d. The vertical and horizontal asymptotes to the graph of f intersect at the point $\mathrm{Q}(1,3)$. *[6 marks]*

Hence find the coordinates of the points on the graph of f that are closest to $(1,3)$.

Markscheme

recognizing that closest is when PQ is a minimum ***(R1)***

eg sketch of PQ, $(\mathrm{PQ})'(x) = 0$

$x = -0.73205$ $x = 2.73205$ (seen anywhere) ***A1A1***

attempt to find *y*-coordinates ***(M1)***

eg $f(-0.73205)$

$(-0.73205, 1.267949), (2.73205, 4.73205)$

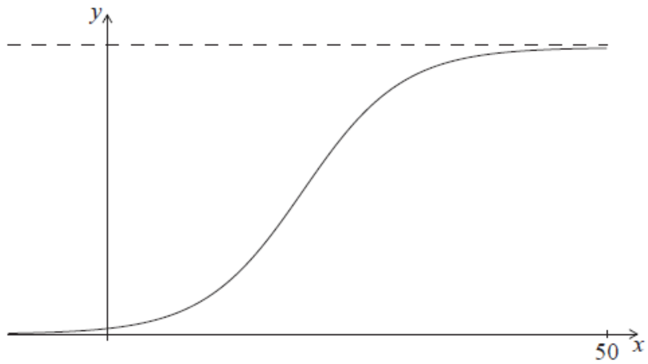
$(-0.732, 1.27), (2.73, 4.73)$ ***A1A1*** ***N4***

[6 marks]

Let

$f(x) = \frac{100}{(1+50e^{-0.2x})}$. Part of the graph of

f is shown below.



7a. Write down $f(0)$. *[1 mark]*

Markscheme

$f(0) = \frac{100}{51}$ (exact), 1.96 ***A1 N1***

[1 mark]

7b. Solve $f(x) = 95$. *[2 marks]*

Markscheme

setting up equation ***(M1)***

eg $95 = \frac{100}{1+50e^{-0.2x}}$, sketch of graph with horizontal line at $y = 95$

$x = 34.3$ ***A1 N2***

[2 marks]

7c. Find the range of f . *[3 marks]*

Markscheme

upper bound of y is 100 ***(A1)***

lower bound of y is 0 ***(A1)***

range is $0 < y < 100$ ***A1 N3***

[3 marks]

7d. Show that $f'(x) = \frac{1000e^{-0.2x}}{(1+50e^{-0.2x})^2}$. *[5 marks]*

Markscheme

METHOD 1

setting function ready to apply the chain rule ***(M1)***

eg $100(1+50\mathrm{e}^{-0.2x})^{-1}$

evidence of correct differentiation (must be substituted into chain rule) ***(A1)(A1)***

eg $u' = -100(1+50\mathrm{e}^{-0.2x})^{-2}$, $v' = (50\mathrm{e}^{-0.2x})(-0.2)$

correct chain rule derivative ***A1***

eg $f'(x) = -100(1+50\mathrm{e}^{-0.2x})^{-2}(50\mathrm{e}^{-0.2x})(-0.2)$

correct working clearly leading to the required answer ***A1***

eg $f'(x) = 1000\mathrm{e}^{-0.2x}(1+50\mathrm{e}^{-0.2x})^{-2}$

$f'(x) = \frac{1000\mathrm{e}^{-0.2x}}{(1+50\mathrm{e}^{-0.2x})^2}$ ***AG N0***

METHOD 2

attempt to apply the quotient rule (accept reversed numerator terms) ***(M1)***

eg $\frac{vu'-uv'}{v^2}$, $\frac{uv'-vu'}{v^2}$

evidence of correct differentiation inside the quotient rule ***(A1)(A1)***

eg $f'(x) = \frac{(1+50\mathrm{e}^{-0.2x})(0)-100(50\mathrm{e}^{-0.2x}\times-0.2)}{(1+50\mathrm{e}^{-0.2x})^2}$, $\frac{100(-10)\mathrm{e}^{-0.2x}-0}{(1+50\mathrm{e}^{-0.2x})^2}$

any correct expression for derivative (0 may not be explicitly seen) ***(A1)***

eg $\frac{-100(50\mathrm{e}^{-0.2x}\times-0.2)}{(1+50\mathrm{e}^{-0.2x})^2}$

correct working clearly leading to the required answer ***A1***

eg $f'(x) = \frac{0-100(-10)\mathrm{e}^{-0.2x}}{(1+50\mathrm{e}^{-0.2x})^2}$, $\frac{-100(-10)\mathrm{e}^{-0.2x}}{(1+50\mathrm{e}^{-0.2x})^2}$

$f'(x) = \frac{1000\mathrm{e}^{-0.2x}}{(1+50\mathrm{e}^{-0.2x})^2}$ ***AG N0***

[5 marks]

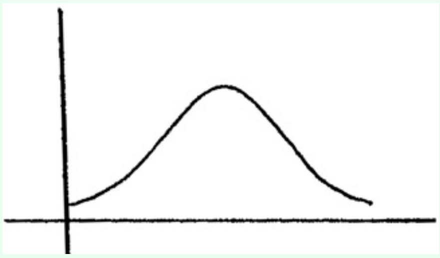
7e. Find the maximum rate of change of f . *[4 marks]*

Markscheme

METHOD 1

sketch of $f'(x)$ ***(A1)***

eg



recognizing maximum on $f'(x)$ ***(M1)***

eg dot on max of sketch

finding maximum on graph of $f'(x)$ ***A1***

eg $(19.6,\ 5)$, $x = 19.560\ldots$

maximum rate of increase is 5 ***A1 N2***

METHOD 2

recognizing $f''(x) = 0$ ***(M1)***

finding any correct expression for $f''(x) = 0$ ***(A1)***

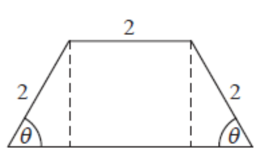
eg $\frac{(1+50e^{-0.2x})^2(-200e^{-0.2x})-(1000e^{-0.2x})(2(1+50e^{-0.2x})(-10e^{-0.2x}))}{(1+50e^{-0.2x})^4}$

finding $x = 19.560\ldots$ ***A1***

maximum rate of increase is 5 ***A1 N2***

[4 marks]

The diagram below shows a plan for a window in the shape of a trapezium.



Three sides of the window are
$2\ \mathrm{m}$ long. The angle between the sloping sides of thewindow and the base is
θ , where
$0 < \theta < \frac{\pi}{2}$.

8a. Show that the area of the window is given by $y = 4\sin\theta + 2\sin 2\theta$. *[5 marks]*

Markscheme

evidence of finding height, *h* ***(A1)***

e.g. $\sin\theta = \frac{h}{2}$, $2\sin\theta$

evidence of finding base of triangle, *b* ***(A1)***

e.g. $\cos\theta = \frac{b}{2}$, $2\cos\theta$

attempt to substitute valid values into a formula for the area of the window ***(M1)***

e.g. two triangles plus rectangle, trapezium area formula

correct expression (must be in terms of θ) ***A1***

e.g. $2\left(\frac{1}{2}\times 2\cos\theta\times 2\sin\theta\right) + 2\times 2\sin\theta$, $\frac{1}{2}(2\sin\theta)(2+2+4\cos\theta)$

attempt to replace $2\sin\theta\cos\theta$ by $\sin 2\theta$ ***M1***

e.g. $4\sin\theta + 2(2\sin\theta\cos\theta)$

$y = 4\sin\theta + 2\sin 2\theta$ ***AG N0***

[5 marks]

8b. Zoe wants a window to have an area of $5\ \mathrm{m}^2$. Find the two possible values of θ . *[4 marks]*

Markscheme

correct equation ***A1***

e.g. $y = 5$, $4\sin\theta + 2\sin 2\theta = 5$

evidence of attempt to solve ***(M1)***

e.g. a sketch, $4\sin\theta + 2\sin\theta - 5 = 0$

$\theta = 0.856\ (49.0^\circ)$, $\theta = 1.25\ (71.4^\circ)$ ***A1A1 N3***

[4 marks]

8c. John wants two windows which have the same area *A* but different values of θ . *[7 marks]*

Find all possible values for *A* .

Markscheme

recognition that lower area value occurs at $\theta = \frac{\pi}{2}$ ***(M1)***

finding value of area at $\theta = \frac{\pi}{2}$ ***(M1)***

e.g. $4\sin\left(\frac{\pi}{2}\right) + 2\sin\left(2 \times \frac{\pi}{2}\right)$, draw square

$A = 4$ ***(A1)***

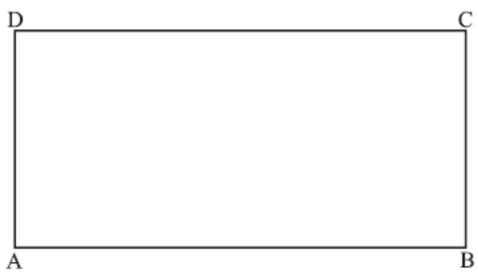
recognition that maximum value of *y* is needed ***(M1)***

$A = 5.19615\ldots$ ***(A1)***

$4 < A < 5.20$ (accept $4 < A < 5.19$) ***A2*** ***N5***

[7 marks]

A farmer wishes to create a rectangular enclosure, ABCD, of area 525 m^2, as shown below.



9. The fencing used for side AB costs $\$11$ per metre. The fencing for the other three sides costs $\$3$ per metre. The farmer creates an enclosure so that the cost is a minimum. Find this minimum cost. *[7 marks]*

Markscheme

METHOD 1

correct expression for **second** side, using area = 525 ***(A1)***

e.g. let $\text{AB} = x$, $\text{AD} = \frac{525}{x}$

attempt to set up cost function using $3 for three sides and $11 for one side ***(M1)***

e.g. $3(\text{AD} + \text{BC} + \text{CD}) + 11\text{AB}$

correct expression for cost ***A2***

e.g. $\frac{525}{x} \times 3 + \frac{525}{x} \times 3 + 11x + 3x$, $\frac{525}{\text{AB}} \times 3 + \frac{525}{\text{AB}} \times 3 + 11\text{AB} + 3\text{AB}$, $\frac{3150}{x} + 14x$

EITHER

sketch of cost function ***(M1)***

identifying minimum point ***(A1)***

e.g. marking point on graph, $x = 15$

minimum cost is 420 (dollars) ***A1*** ***N4***

OR

correct derivative (may be seen in equation below) ***(A1)***

e.g. $C'(x) = \frac{-1575}{x^2} + \frac{-1575}{x^2} + 14$

setting their derivative equal to 0 (seen anywhere) ***(M1)***

e.g. $\frac{-3150}{x^2} + 14 = 0$

minimum cost is 420 (dollars) ***A1*** ***N4***

METHOD 2

correct expression for **second** side, using area = 525 ***(A1)***

e.g. let $\mathrm{AD} = x$, $\mathrm{AB} = \frac{525}{x}$

attempt to set up cost function using $\$3$ for three sides and $\$11$ for one side ***(M1)***

e.g. $3(\mathrm{AD} + \mathrm{BC} + \mathrm{CD}) + 11\mathrm{AB}$

correct expression for cost ***A2***

e.g. $3\left(x + x + \frac{525}{x}\right) + \frac{525}{x} \times 11$, $3\left(\mathrm{AD} + \mathrm{AD} + \frac{525}{\mathrm{AD}}\right) + \frac{525}{\mathrm{AD}} \times 11$, $6x + \frac{7350}{x}$

EITHER

sketch of cost function ***(M1)***

identifying minimum point ***(A1)***

e.g. marking point on graph, $x = 35$

minimum cost is 420 (dollars) ***A1*** ***N4***

OR

correct derivative (may be seen in equation below) ***(A1)***

e.g. $C'(x) = 6 - \frac{7350}{x^2}$

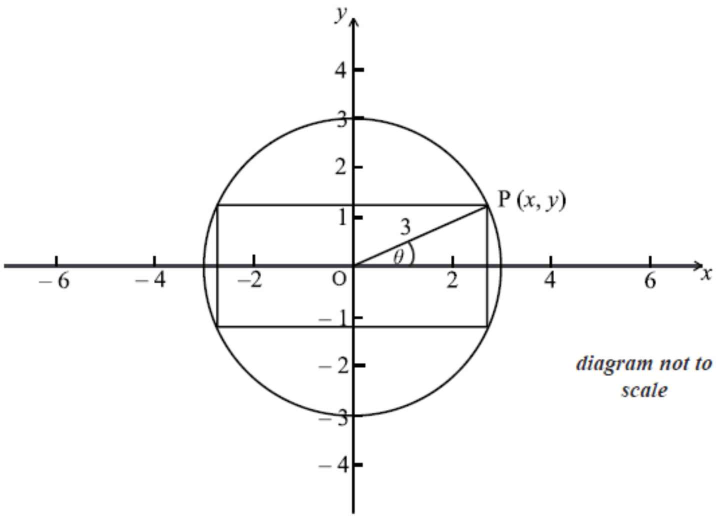
setting their derivative equal to 0 (seen anywhere) ***(M1)***

e.g. $6 - \frac{7350}{x^2} = 0$

minimum cost is 420 (dollars) ***A1*** ***N4***

[7 marks]

A rectangle is inscribed in a circle of radius 3 cm and centre O, as shown below.



The point P(x , y) is a vertex of the rectangle and also lies on the circle. The anglebetween (OP) and the x-axis is θ radians, where
$0 \le \theta \le \frac{\pi}{2}$.

10a. Write down an expression in terms of θ for *[2 marks]*

(i) x ;

(ii) y .

Markscheme

(i) $x = 3\cos\theta$ ***A1 N1***

(ii) $y = 3\sin\theta$ ***A1 N1***

[2 marks]

10b. Let the area of the rectangle be A. *[3 marks]*

Show that $A = 18\sin 2\theta$.

Markscheme

finding area ***(M1)***

e.g. $A = 2x \times 2y$, $A = 8 \times \frac{1}{2}bh$

substituting ***A1***

e.g. $A = 4 \times 3\sin\theta \times 3\cos\theta$, $8 \times \frac{1}{2} \times 3\cos\theta \times 3\sin\theta$

$A = 18(2\sin\theta\cos\theta)$ ***A1***

$A = 18\sin 2\theta$ ***AG N0***

[3 marks]

10c. (i) Find $\frac{\mathrm{d}A}{\mathrm{d}\theta}$. *[8 marks]*

(ii) Hence, find the exact value of θ which maximizes the area of the rectangle.

(iii) Use the second derivative to justify that this value of θ does give a maximum.

Markscheme

(i) $\frac{\mathrm{d}A}{\mathrm{d}\theta} = 36\cos 2\theta$ ***A2 N2***

(ii) for setting derivative equal to 0 ***(M1)***

e.g. $36\cos 2\theta = 0$, $\frac{\mathrm{d}A}{\mathrm{d}\theta} = 0$

$2\theta = \frac{\pi}{2}$ ***(A1)***

$\theta = \frac{\pi}{4}$ ***A1 N2***

(iii) valid reason (seen anywhere) ***R1***

e.g. at $\frac{\pi}{4}$, $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} < 0$; maximum when $f''(x) < 0$

finding second derivative $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = -72\sin 2\theta$ ***A1***

evidence of substituting $\frac{\pi}{4}$ ***M1***

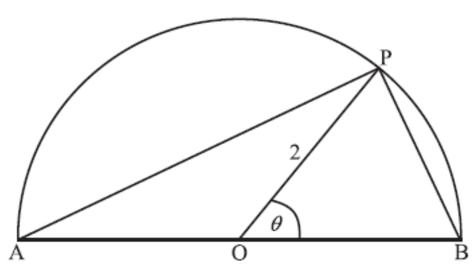
e.g. $-72\sin\left(2 \times \frac{\pi}{4}\right)$, $-72\sin\left(\frac{\pi}{2}\right)$, -72

$\theta = \frac{\pi}{4}$ produces the maximum area ***AG N0***

[8 marks]

The following diagram shows a semicircle centre O, diameter [AB], with radius 2.

Let P be a point on the circumference, with $\mathrm{P\widehat{O}B} = \theta$ radians.



11a. Find the area of the triangle OPB, in terms of θ . *[2 marks]*

Markscheme

evidence of using area of a triangle ***(M1)***

e.g. $A = \frac{1}{2} \times 2 \times 2 \times \sin\theta$

$A = 2\sin\theta$ ***A1 N2***

[2 marks]

11b. Explain why the area of triangle OPA is the same as the area triangle OPB. *[3 marks]*

Markscheme

METHOD 1

$\text{P}\hat{\text{O}}\text{A} = \pi - \theta$ ***(A1)***

area $\Delta\text{OPA} = \frac{1}{2} 2 \times 2 \times \sin(\pi - \theta)\ (= 2\sin(\pi - \theta))$ ***A1***

since $\sin(\pi - \theta) = \sin\theta$ ***R1***

then both triangles have the same area ***AG*** ***N0***

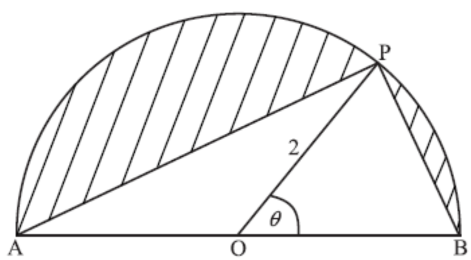
METHOD 2

triangle OPA has the same height and the same base as triangle OPB ***R3***

then both triangles have the same area ***AG*** ***N0***

[3 marks]

Let S be the total area of the two segments shaded in the diagram below.



11c. Show that $S = 2(\pi - 2\sin\theta)$. *[3 marks]*

Markscheme

area semicircle $= \frac{1}{2} \times \pi(2)^2\ (= 2\pi)$ ***A1***

area $\Delta\text{APB} = 2\sin\theta + 2\sin\theta\ (= 4\sin\theta)$ ***A1***

$S = \text{area of semicircle} - \text{area}\ \Delta\text{APB}\ (= 2\pi - 4\sin\theta)$ ***M1***

$S = 2(\pi - 2\sin\theta)$ ***AG*** ***N0***

[3 marks]

11d. Find the value of θ when S is a local minimum, justifying that it is a minimum. *[8 marks]*

Markscheme

METHOD 1

attempt to differentiate ***(M1)***

e.g. $\frac{\mathrm{d}S}{\mathrm{d}\theta} = -4\cos\theta$

setting derivative equal to 0 ***(M1)***

correct equation ***A1***

e.g. $-4\cos\theta = 0$, $\cos\theta = 0$, $4\cos\theta = 0$

$\theta = \frac{\pi}{2}$ ***A1*** ***N3***

EITHER

evidence of using second derivative ***(M1)***

$S''(\theta) = 4\sin\theta$ ***A1***

$S''\left(\frac{\pi}{2}\right) = 4$ ***A1***

it is a minimum because $S''\left(\frac{\pi}{2}\right) > 0$ ***R1*** ***N0***

OR

evidence of using first derivative ***(M1)***

for $\theta < \frac{\pi}{2}, S'(\theta) < 0$ (may use diagram) ***A1***

for $\theta > \frac{\pi}{2}, S'(\theta) > 0$ (may use diagram) ***A1***

it is a minimum since the derivative goes from negative to positive ***R1*** ***N0***

METHOD 2

$2\pi - 4\sin\theta$ is minimum when $4\sin\theta$ is a maximum ***R3***

$4\sin\theta$ is a maximum when $\sin\theta = 1$ ***(A2)***

$\theta = \frac{\pi}{2}$ ***A3*** ***N3***

[8 marks]

11e. Find a value of θ for which S has its greatest value. *[2 marks]*

Markscheme

S is greatest when $4\sin\theta$ is smallest (or equivalent) ***(R1)***

$\theta = 0$ (or π) ***A1*** ***N2***

[2 marks]