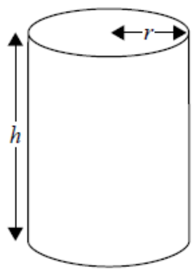
Optimization *[153 marks]*

A closed cylindrical can with radius r centimetres and height h centimetres has a volume of 20 π cm^3.

diagram not to scale



1a. Express *h* in terms of *r*. *[2 marks]*

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The material for the base and top of the can costs 10 cents per cm ^{2} and the material for the curved side costs 8 cents per cm^2. The total cost of the material, in cents, is C.

1b. Show that $C = 20\pi r^2 + \frac{320\pi}{r}$. *[4 marks]*

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1c. Given that there is a minimum value for C, find this minimum value in terms of π. *[9 marks]*

2. Consider $f(x) = \log k(6x - 3x^2)$, for $0 < x < 2$, where $k > 0$. *[7 marks]*

The equation $f(x) = 2$ has exactly one solution. Find the value of k.

Let $f(x) = \ln x$ and $g(x) = 3 + \ln\left(\frac{x}{2}\right)$, for $x > 0$.

The graph of g can be obtained from the graph of f by two transformations:

a horizontal stretch of scale factor q followed by

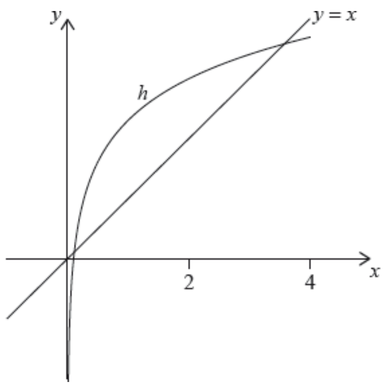
a translation of $\begin{pmatrix} h \\ k \end{pmatrix}$.

3a. Write down the value of q; *[1 mark]*

3b. Write down the value of h; *[1 mark]*

3c. Write down the value of k. *[1 mark]*

Let $h(x) = g(x) \times \cos(0.1x)$, for $0 < x < 4$. The following diagram shows the graph of h and the line $y = x$.



The graph of h intersects the graph of h^{-1} at two points. These points have x coordinates 0.111 and 3.31 correct to three significant figures.

3d. Find $\int_{0.111}^{3.31} (h(x) - x)\,\mathrm{d}x$. *[2 marks]*

3e. Hence, find the area of the region enclosed by the graphs of h and h^{-1}. *[3 marks]*

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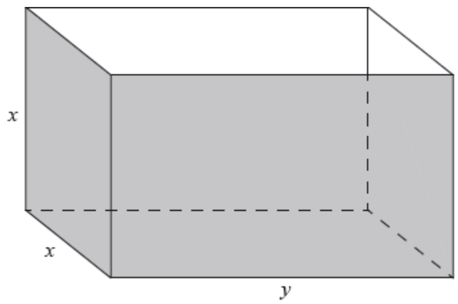
3f. Let d be the vertical distance from a point on the graph of h to the line $y = x$. There is a point $\mathrm{P}(a,\ b)$ on the graph of h where d is a maximum. *[7 marks]*

Find the coordinates of P, where $0.111 < a < 3.31$.

Fred makes an open metal container in the shape of a cuboid, as shown in the following diagram.



The container has height x m, width x m and length y m. The volume is 36 m^3.

Let $A(x)$ be the outside surface area of the container.

4a. Show that $A(x) = \frac{108}{x} + 2x^2$. *[4 marks]*

4b. Find $A'(x)$. *[2 marks]*

4c. Given that the outside surface area is a minimum, find the height of the container. *[5 marks]*

4d. Fred paints the outside of the container. A tin of paint covers a surface area of $10\ \mathrm{m}^2$ and costs $20. Find the total cost of the tins needed to paint the container. *[5 marks]*

Let L_x be a family of lines with equation given by $r = \begin{pmatrix} x \\ \frac{2}{x} \end{pmatrix} + t\begin{pmatrix} x^2 \\ -2 \end{pmatrix}$, where $x > 0$.

5a. Write down the equation of L_1. *[2 marks]*

5b. A line L_a crosses the y-axis at a point P. *[6 marks]*

Show that P has coordinates $\left(0,\ \frac{4}{a}\right)$.

5c. The line L_a crosses the x-axis at $\mathrm{Q}(2a,\ 0)$. Let $d = \mathrm{PQ}^2$. *[2 marks]*

Show that $d = 4a^2 + \frac{16}{a^2}$.

5d. There is a minimum value for d. Find the value of a that gives this minimum value. *[7 marks]*

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Let
$f(x) = \frac{3x}{x-q}$, where
$x \neq q$.

6a. Write down the equations of the vertical and horizontal asymptotes of the graph of f. *[2 marks]*

6b. The vertical and horizontal asymptotes to the graph of f intersect at the point $\mathrm{Q}(1, 3)$. *[2 marks]*

Find the value of q.

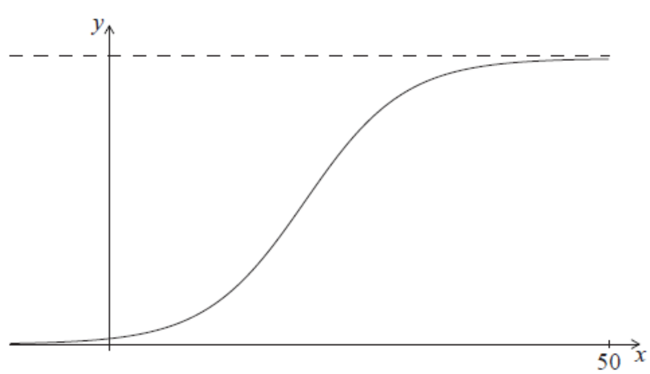
6c. The vertical and horizontal asymptotes to the graph of f intersect at the point $\mathrm{Q}(1, 3)$. *[4 marks]*

The point $\mathrm{P}(x,\ y)$ lies on the graph of f. Show that $\mathrm{PQ} = \sqrt{(x-1)^2 + \left(\frac{3}{x-1}\right)^2}$.

6d. The vertical and horizontal asymptotes to the graph of f intersect at the point $\mathrm{Q}(1, 3)$. *[6 marks]*

Hence find the coordinates of the points on the graph of f that are closest to $(1, 3)$.

Let
$f(x) = \frac{100}{(1+50\mathrm{e}^{-0.2x})}$. Part of the graph of
f is shown below.



7a. Write down $f(0)$. *[1 mark]*

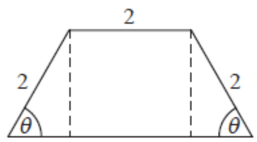
7b. Solve $f(x) = 95$. *[2 marks]*

7c. Find the range of f. *[3 marks]*

7d. Show that $f'(x) = \frac{1000e^{-0.2x}}{(1+50e^{-0.2x})^2}$. *[5 marks]*

7e. Find the maximum rate of change of f. *[4 marks]*

The diagram below shows a plan for a window in the shape of a trapezium.



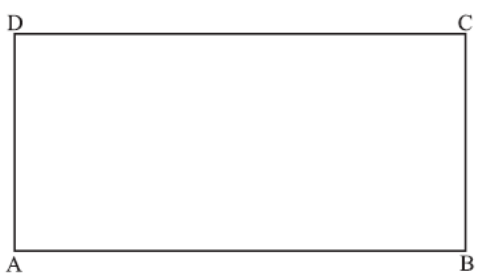
Three sides of the window are
2 m long. The angle between the sloping sides of thewindow and the base is
θ, where
$0 < \theta < \frac{\pi}{2}$.

8a. Show that the area of the window is given by $y = 4\sin\theta + 2\sin 2\theta$. *[5 marks]*

8b. Zoe wants a window to have an area of 5 m^2. Find the two possible values of θ. *[4 marks]*

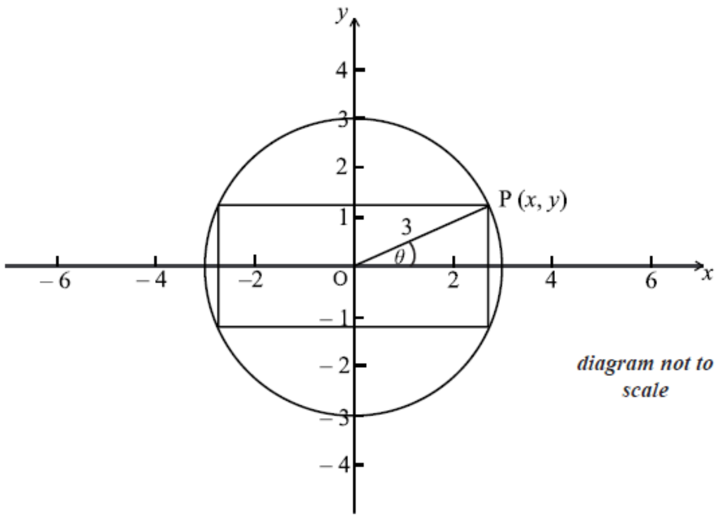
8c. John wants two windows which have the same area *A* but different values of θ. Find all possible values for *A*. *[7 marks]*

A farmer wishes to create a rectangular enclosure, ABCD, of area 525 m^2, as shown below.



9. The fencing used for side AB costs $\$11$ per metre. The fencing for the other three sides costs $\$3$ per metre. The farmer creates an enclosure so that the cost is a minimum. Find this minimum cost. *[7 marks]*

A rectangle is inscribed in a circle of radius 3 cm and centre O, as shown below.



The point P(*x* , *y*) is a vertex of the rectangle and also lies on the circle. The anglebetween (OP) and the *x*-axis is θ radians, where
$0 \le \theta \le \frac{\pi}{2}$.

10a. Write down an expression in terms of θ for *[2 marks]*

(i) x ;

(ii) y .

10b. Let the area of the rectangle be *A*. *[3 marks]*

Show that $A = 18\sin 2\theta$.

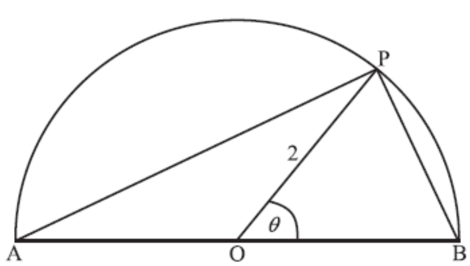
10c. (i) Find $\frac{\mathrm{d}A}{\mathrm{d}\theta}$. *[8 marks]*

(ii) Hence, find the exact value of θ which maximizes the area of the rectangle.

(iii) Use the second derivative to justify that this value of θ does give a maximum.

The following diagram shows a semicircle centre O, diameter [AB], with radius 2.

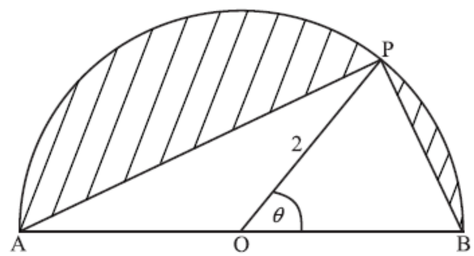
Let P be a point on the circumference, with
$\mathrm{P\hat{O}B} = \theta$ radians.



11a. Find the area of the triangle OPB, in terms of θ . *[2 marks]*

11b. Explain why the area of triangle OPA is the same as the area triangle OPB. *[3 marks]*

Let S be the total area of the two segments shaded in the diagram below.



11c. Show that $S = 2(\pi - 2\sin\theta)$. *[3 marks]*

11d. Find the value of θ when S is a local minimum, justifying that it is a minimum. *[8 marks]*

11e. Find a value of θ for which S has its greatest value. *[2 marks]*