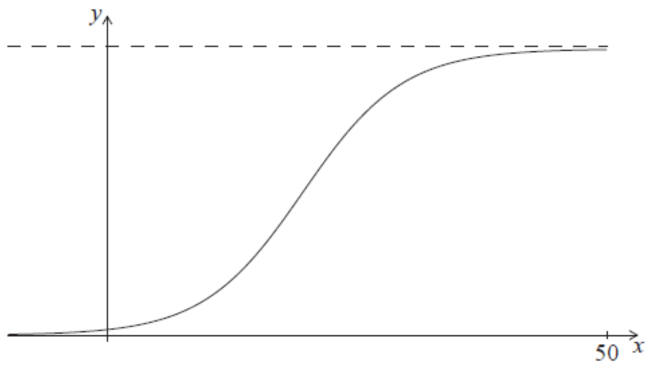
Last Drivatives *[124 marks]*

Let
$f(x) = \frac{100}{(1+50e^{-0.2x})}$. Part of the graph of
f is shown below.



1a. Write down $f(0)$. *[1 mark]*

Markscheme

$f(0) = \frac{100}{51}$ (exact), 1.96 ***A1*** ***N1***

[1 mark]

1b. Solve $f(x) = 95$. *[2 marks]*

Markscheme

setting up equation ***(M1)***

eg $95 = \frac{100}{1+50e^{-0.2x}}$, sketch of graph with horizontal line at $y = 95$

$x = 34.3$ ***A1*** ***N2***

[2 marks]

1c. Find the range of f . *[3 marks]*

Markscheme

upper bound of y is 100 ***(A1)***

lower bound of y is 0 ***(A1)***

range is $0 < y < 100$ ***A1*** ***N3***

[3 marks]

1d. Show that $f'(x) = \frac{1000\mathrm{e}^{-0.2x}}{(1+50\mathrm{e}^{-0.2x})^2}$. *[5 marks]*

Markscheme

METHOD 1

setting function ready to apply the chain rule ***(M1)***

eg $100(1+50\mathrm{e}^{-0.2x})^{-1}$

evidence of correct differentiation (must be substituted into chain rule) ***(A1)(A1)***

eg $u' = -100(1+50\mathrm{e}^{-0.2x})^{-2}$, $v' = (50\mathrm{e}^{-0.2x})(-0.2)$

correct chain rule derivative ***A1***

eg $f'(x) = -100(1+50\mathrm{e}^{-0.2x})^{-2}(50\mathrm{e}^{-0.2x})(-0.2)$

correct working clearly leading to the required answer ***A1***

eg $f'(x) = 1000\mathrm{e}^{-0.2x}(1+50\mathrm{e}^{-0.2x})^{-2}$

$f'(x) = \frac{1000\mathrm{e}^{-0.2x}}{(1+50\mathrm{e}^{-0.2x})^2}$ ***AG*** ***N0***

METHOD 2

attempt to apply the quotient rule (accept reversed numerator terms) ***(M1)***

eg $\frac{vu'-uv'}{v^2}$, $\frac{uv'-vu'}{v^2}$

evidence of correct differentiation inside the quotient rule ***(A1)(A1)***

eg $f'(x) = \frac{(1+50\mathrm{e}^{-0.2x})(0)-100(50\mathrm{e}^{-0.2x}\times-0.2)}{(1+50\mathrm{e}^{-0.2x})^2}$, $\frac{100(-10)\mathrm{e}^{-0.2x}-0}{(1+50\mathrm{e}^{-0.2x})^2}$

any correct expression for derivative (0 may not be explicitly seen) ***(A1)***

eg $\frac{-100(50\mathrm{e}^{-0.2x}\times-0.2)}{(1+50\mathrm{e}^{-0.2x})^2}$

correct working clearly leading to the required answer ***A1***

eg $f'(x) = \frac{0-100(-10)\mathrm{e}^{-0.2x}}{(1+50\mathrm{e}^{-0.2x})^2}$, $\frac{-100(-10)\mathrm{e}^{-0.2x}}{(1+50\mathrm{e}^{-0.2x})^2}$

$f'(x) = \frac{1000\mathrm{e}^{-0.2x}}{(1+50\mathrm{e}^{-0.2x})^2}$ ***AG*** ***N0***

[5 marks]

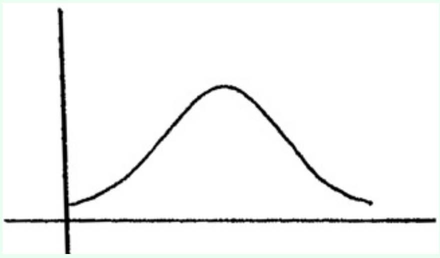
1e. Find the maximum rate of change of f . *[4 marks]*

Markscheme

METHOD 1

sketch of $f'(x)$ ***(A1)***

eg



recognizing maximum on $f'(x)$ ***(M1)***

eg dot on max of sketch

finding maximum on graph of $f'(x)$ ***A1***

eg $(19.6,\ 5)$, $x = 19.560\ldots$

maximum rate of increase is 5 ***A1 N2***

METHOD 2

recognizing $f''(x) = 0$ ***(M1)***

finding any correct expression for $f''(x) = 0$ ***(A1)***

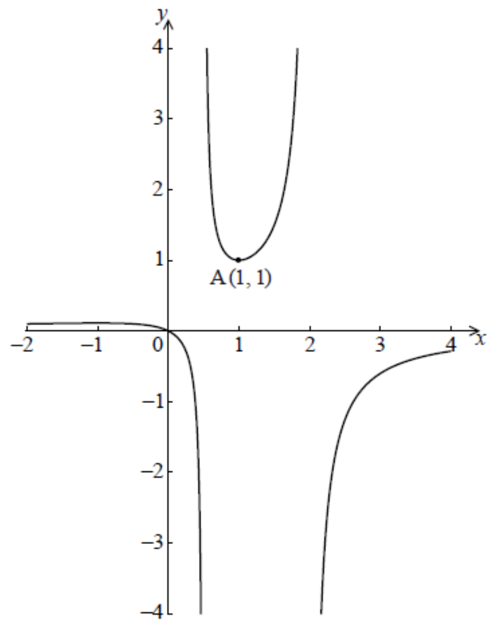
eg $\frac{(1+50\mathrm{e}^{-0.2x})^2(-200\mathrm{e}^{-0.2x})-(1000\mathrm{e}^{-0.2x})(2(1+50\mathrm{e}^{-0.2x})(-10\mathrm{e}^{-0.2x}))}{(1+50\mathrm{e}^{-0.2x})^4}$

finding $x = 19.560\ldots$ ***A1***

maximum rate of increase is 5 ***A1*** ***N2***

[4 marks]

Let
$f(x) = \frac{x}{-2x^2+5x-2}$ for
$-2 \le x \le 4$,
$x \ne \frac{1}{2}$,
$x \ne 2$. The graph of
f is given below.



The graph of
f has a local minimum at A(
1,
1) and a local maximum at B.

2a. Use the quotient rule to show that $f'(x) = \frac{2x^2-2}{(-2x^2+5x-2)^2}$. *[6 marks]*

Markscheme

correct derivatives **applied** in quotient rule ***(A1)A1A1***

$1,\ -4x+5$

Note: Award ***(A1)*** for 1, ***A1*** for $-4x$ and ***A1*** for 5, **only** if it is clear candidates are using the quotient rule.

correct substitution into quotient rule ***A1***

e.g. $\frac{1\times(-2x^2+5x-2)-x(-4x+5)}{(-2x^2+5x-2)^2}$, $\frac{-2x^2+5x-2-x(-4x+5)}{(-2x^2+5x-2)^2}$

correct working ***(A1)***

e.g. $\frac{-2x^2+5x-2-(-4x^2+5x)}{(-2x^2+5x-2)^2}$

expression clearly leading to the answer ***A1***

e.g. $\frac{-2x^2+5x-2+4x^2-5x}{(-2x^2+5x-2)^2}$

$f'(x)=\frac{2x^2-2}{(-2x^2+5x-2)^2}$ ***AG N0***

[6 marks]

2b. Hence find the coordinates of B. *[7 marks]*

Markscheme

evidence of attempting to solve $f'(x)=0$ ***(M1)***

e.g. $2x^2-2=0$

evidence of correct working ***A1***

e.g. $x^2=1, \frac{\pm\sqrt{16}}{4}, 2(x-1)(x+1)$

correct solution to quadratic ***(A1)***

e.g. $x=\pm 1$

correct *x*-coordinate $x=-1$ (may be seen in coordinate form $\left(-1,\frac{1}{9}\right)$) ***A1 N2***

attempt to substitute -1 into *f* (do not accept any other value) ***(M1)***

e.g. $f(-1)=\frac{-1}{-2\times(-1)^2+5\times(-1)-2}$

correct working

e.g. $\frac{-1}{-2-5-2}$ ***A1***

correct *y*-coordinate $y=\frac{1}{9}$ (may be seen in coordinate form $\left(-1,\frac{1}{9}\right)$) ***A1 N2***

[7 marks]

2c. Given that the line $y=k$ does not meet the graph of *f* , find the possible values of *k* . *[3 marks]*

Markscheme

recognizing values between max and min ***(R1)***

$\frac{1}{9} < k < 1$ ***A2*** ***N3***

[3 marks]

Let
$g(x) = \frac{\ln x}{x^2}$, for
$x > 0$.

3a. Use the quotient rule to show that $g'(x) = \frac{1-2\ln x}{x^3}$. *[4 marks]*

Markscheme

$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$, $\frac{\mathrm{d}}{\mathrm{d}x}x^2 = 2x$ (seen anywhere) ***A1A1***

attempt to substitute into the quotient rule (do **not** accept product rule) ***M1***

e.g. $\frac{x^2\left(\frac{1}{x}\right)-2x\ln x}{x^4}$

correct manipulation that clearly leads to result ***A1***

e.g. $\frac{x-2x\ln x}{x^4}$, $\frac{x(1-2\ln x)}{x^4}$, $\frac{x}{x^4}$, $\frac{2x\ln x}{x^4}$

$g'(x) = \frac{1-2\ln x}{x^3}$ ***AG*** ***N0***

[4 marks]

3b. The graph of *g* has a maximum point at A. Find the *x*-coordinate of A. *[3 marks]*

Markscheme

evidence of setting the derivative equal to zero ***(M1)***

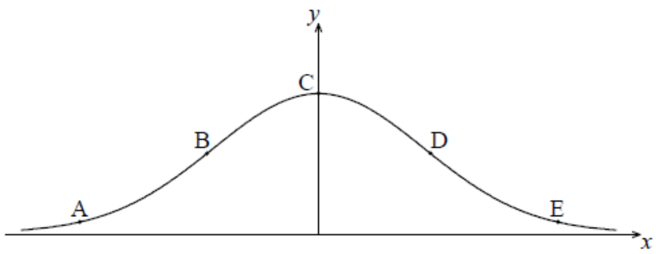
e.g. $g'(x) = 0$, $1 - 2\ln x = 0$

$\ln x = \frac{1}{2}$ ***A1***

$x = \mathrm{e}^{\frac{1}{2}}$ ***A1*** ***N2***

[3 marks]

The following diagram shows the graph of
$f(x) = \mathrm{e}^{-x^2}$.



The points A, B, C, D and E lie on the graph of f . Two of these are points of inflexion.

4a. Identify the **two** points of inflexion. *[2 marks]*

Markscheme

B, D ***A1A1*** ***N2***

[2 marks]

4b. (i) Find $f'(x)$. *[5 marks]*

(ii) Show that $f''(x) = (4x^2 - 2)\mathrm{e}^{-x^2}$.

Markscheme

(i) $f'(x) = -2x\mathrm{e}^{-x^2}$ ***A1A1*** ***N2***

Note: Award ***A1*** for e^{-x^2} and ***A1*** for $-2x$.

(ii) finding the derivative of $-2x$, i.e. -2 ***(A1)***

evidence of choosing the product rule ***(M1)***

e.g. $-2\mathrm{e}^{-x^2} - 2x \times -2x\mathrm{e}^{-x^2}$

$-2\mathrm{e}^{-x^2} + 4x^2\mathrm{e}^{-x^2}$ ***A1***

$f''(x) = (4x^2 - 2)\mathrm{e}^{-x^2}$ ***AG*** ***N0***

[5 marks]

4c. Find the x-coordinate of each point of inflexion. *[4 marks]*

Markscheme

valid reasoning ***R1***

e.g. $f''(x) = 0$

attempting to solve the equation ***(M1)***

e.g. $(4x^2 - 2) = 0$, sketch of $f''(x)$

$p = 0.707 \left(= \frac{1}{\sqrt{2}}\right)$, $q = -0.707 \left(= -\frac{1}{\sqrt{2}}\right)$ ***A1A1*** ***N3***

[4 marks]

4d. Use the second derivative to show that one of these points is a point of inflexion. *[4 marks]*

Markscheme

evidence of using second derivative to test values on either side of POI ***M1***

e.g. finding values, reference to graph of f'' , sign table

correct working ***A1A1***

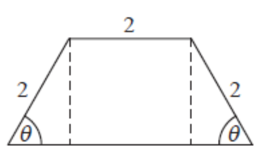
e.g. finding any two correct values either side of POI,

checking sign of f'' on either side of POI

reference to sign change of $f''(x)$ ***R1*** ***N0***

[4 marks]

The diagram below shows a plan for a window in the shape of a trapezium.



Three sides of the window are
2 m long. The angle between the sloping sides of thewindow and the base is
θ , where
$0 < \theta < \frac{\pi}{2}$.

5a. Show that the area of the window is given by $y = 4\sin\theta + 2\sin 2\theta$. *[5 marks]*

Markscheme

evidence of finding height, *h* ***(A1)***

e.g. $\sin\theta = \frac{h}{2}$, $2\sin\theta$

evidence of finding base of triangle, *b* ***(A1)***

e.g. $\cos\theta = \frac{b}{2}$, $2\cos\theta$

attempt to substitute valid values into a formula for the area of the window ***(M1)***

e.g. two triangles plus rectangle, trapezium area formula

correct expression (must be in terms of θ) ***A1***

e.g. $2\left(\frac{1}{2} \times 2\cos\theta \times 2\sin\theta\right) + 2 \times 2\sin\theta$, $\frac{1}{2}(2\sin\theta)(2 + 2 + 4\cos\theta)$

attempt to replace $2\sin\theta\cos\theta$ by $\sin 2\theta$ ***M1***

e.g. $4\sin\theta + 2(2\sin\theta\cos\theta)$

$y = 4\sin\theta + 2\sin 2\theta$ ***AG*** ***N0***

[5 marks]

5b. Zoe wants a window to have an area of $5\ \mathrm{m}^2$. Find the two possible values of θ . *[4 marks]*

Markscheme

correct equation ***A1***

e.g. $y = 5$, $4\sin\theta + 2\sin 2\theta = 5$

evidence of attempt to solve ***(M1)***

e.g. a sketch, $4\sin\theta + 2\sin\theta - 5 = 0$

$\theta = 0.856\ (49.0^\circ)$, $\theta = 1.25\ (71.4^\circ)$ ***A1A1*** ***N3***

[4 marks]

5c. John wants two windows which have the same area *A* but different values of θ . *[7 marks]*

Find all possible values for *A* .

Markscheme

recognition that lower area value occurs at $\theta = \frac{\pi}{2}$ ***(M1)***

finding value of area at $\theta = \frac{\pi}{2}$ ***(M1)***

e.g. $4\sin\left(\frac{\pi}{2}\right) + 2\sin\left(2 \times \frac{\pi}{2}\right)$, draw square

$A = 4$ ***(A1)***

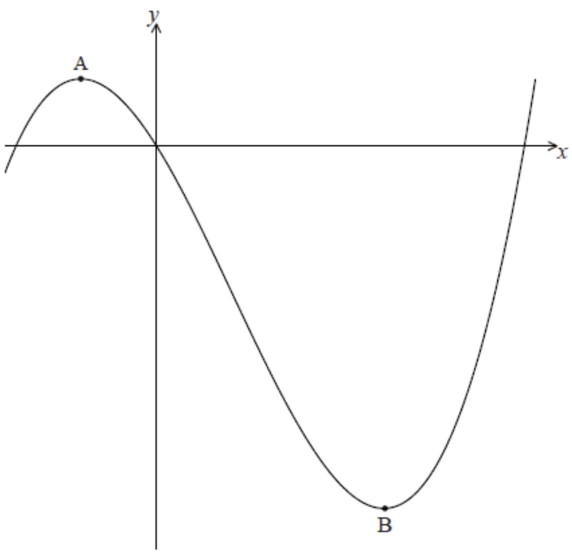
recognition that maximum value of y is needed ***(M1)***

$A = 5.19615\ldots$ ***(A1)***

$4 < A < 5.20$ (accept $4 < A < 5.19$) ***A2*** ***N5***

[7 marks]

Let
$f(x) = \frac{1}{2}x^3 - x^2 - 3x$. Part of the graph of f is shown below.



There is a maximum point at A and a minimum point at B(3, − 9) .

6a. Find the coordinates of A. *[8 marks]*

Markscheme

$f(x) = x^2 - 2x - 3$ ***A1A1A1***

evidence of solving $f'(x) = 0$ ***(M1)***

e.g. $x^2 - 2x - 3 = 0$

evidence of correct working ***A1***

e.g. $(x+1)(x-3)$, $\frac{2 \pm \sqrt{16}}{2}$

$x = -1$ (ignore $x = 3$) ***(A1)***

evidence of substituting **their** negative *x*-value into $f(x)$ ***(M1)***

e.g. $\frac{1}{3}(-1)^3 - (-1)^2 - 3(-1)$, $-\frac{1}{3} - 1 + 3$

$y = \frac{5}{3}$ ***A1***

coordinates are $\left(-1, \frac{5}{3}\right)$ ***N3***

[8 marks]

6b. Write down the coordinates of *[6 marks]*

(i) the image of B after reflection in the *y*-axis;

(ii) the image of B after translation by the vector $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$;

(iii) the image of B after reflection in the *x*-axis followed by a horizontal stretch with scale factor $\frac{1}{2}$.

Markscheme

(i) $(-3, -9)$ ***A1 N1***

(ii) $(1, -4)$ ***A1A1 N2***

(iii) reflection gives $(3, 9)$ ***(A1)***

stretch gives $\left(\frac{3}{2}, 9\right)$ ***A1A1 N3***

[6 marks]

Let
$f(x) = \frac{\cos x}{\sin x}$, for
$\sin x \neq 0$.

7a. Use the quotient rule to show that $f'(x) = \frac{-1}{\sin^2 x}$. *[5 marks]*

Markscheme

$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$, $\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$ (seen anywhere) ***(A1)(A1)***

evidence of using the quotient rule ***M1***

correct substitution ***A1***

e.g. $\frac{\sin x(-\sin x)-\cos x(\cos x)}{\sin^2 x}$, $\frac{-\sin^2 x-\cos^2 x}{\sin^2 x}$

$f'(x) = \frac{-(\sin^2 x+\cos^2 x)}{\sin^2 x}$ ***A1***

$f'(x) = \frac{-1}{\sin^2 x}$ ***AG*** ***N0***

[5 marks]

7b. Find $f''(x)$. *[3 marks]*

Markscheme

METHOD 1

appropriate approach ***(M1)***

e.g. $f'(x) = -(\sin x)^{-2}$

$f''(x) = 2(\sin^{-3} x)(\cos x) \left(= \frac{2\cos x}{\sin^3 x}\right)$ ***A1A1*** ***N3***

Note: Award ***A1*** for $2\sin^{-3} x$, ***A1*** for $\cos x$.

METHOD 2

derivative of $\sin^2 x = 2\sin x\cos x$ (seen anywhere) ***A1***

evidence of choosing quotient rule ***(M1)***

e.g. $u = -1$, $v = \sin^2 x$, $f'' = \frac{\sin^2 x\times 0-(-1)2\sin x\cos x}{(\sin^2 x)^2}$

$f''(x) = \frac{2\sin x\cos x}{(\sin^2 x)^2} \left(= \frac{2\cos x}{\sin^3 x}\right)$ ***A1*** ***N3***

[3 marks]

In the following table,
$f'\left(\frac{\pi}{2}\right) = p$ and
$f''\left(\frac{\pi}{2}\right) = q$. The table also gives approximate values of
$f'(x)$ and
$f''(x)$ near
$x = \frac{\pi}{2}$.

x	$\frac{\pi}{2}-0.1$	$\frac{\pi}{2}$	$\frac{\pi}{2}+0.1$
$f'(x)$	-1.01	p	-1.01
$f''(x)$	0.203	q	-0.203

7c. Find the value of *p* and of *q*. *[3 marks]*

Markscheme

evidence of substituting $\frac{\pi}{2}$ ***M1***

e.g. $\frac{-1}{\sin^2\frac{\pi}{2}}$, $\frac{2\cos\frac{\pi}{2}}{\sin^3\frac{\pi}{2}}$

$p = -1$, $q = 0$ ***A1A1*** ***N1N1***

[3 marks]

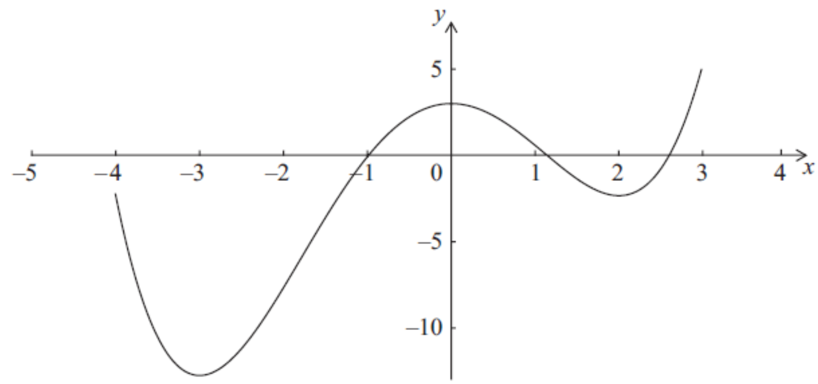
7d. Use information from the table to explain why there is a point of inflexion on the graph of *f* where $x = \frac{\pi}{2}$. *[2 marks]*

Markscheme

second derivative is zero, second derivative changes sign ***R1R1*** ***N2***

[2 marks]

A function *f* is defined for $-4 \le x \le 3$. The graph of *f* is given below.



The graph has a local maximum when $x = 0$, and local minima when $x = -3$, $x = 2$.

8a. Write down the *x*-intercepts of the graph of the **derivative** function, f' . *[2 marks]*

Markscheme

x-intercepts at -3, 0, 2 ***A2 N2***

[2 marks]

8b. Write down all values of *x* for which $f'(x)$ is positive. *[2 marks]*

Markscheme

$-3 < x < 0$, $2 < x < 3$ ***A1A1 N2***

[2 marks]

8c. At point D on the graph of *f* , the *x*-coordinate is -0.5. Explain why $f''(x) < 0$ at D. *[2 marks]*

Markscheme

correct reasoning ***R2***

e.g. the graph of *f* is **concave-down** (accept convex), the first derivative is decreasing therefore the second derivative is negative ***AG***

[2 marks]

Consider
$f(x) = x^2 + \frac{p}{x}$,
$x \neq 0$, where *p* is a constant.

9a. Find $f'(x)$. *[2 marks]*

Markscheme

$f'(x) = 2x - \frac{p}{x^2}$ ***A1A1*** ***N2***

Note: Award ***A1*** for $2x$, ***A1*** for $-\frac{p}{x^2}$.

[2 marks]

9b. There is a minimum value of $f(x)$ when $x = -2$. Find the value of p . *[4 marks]*

Markscheme

evidence of equating derivative to 0 (seen anywhere) ***(M1)***

evidence of finding $f'(-2)$ (seen anywhere) ***(M1)***

correct equation ***A1***

e.g. $-4 - \frac{p}{4} = 0$, $-16 - p = 0$

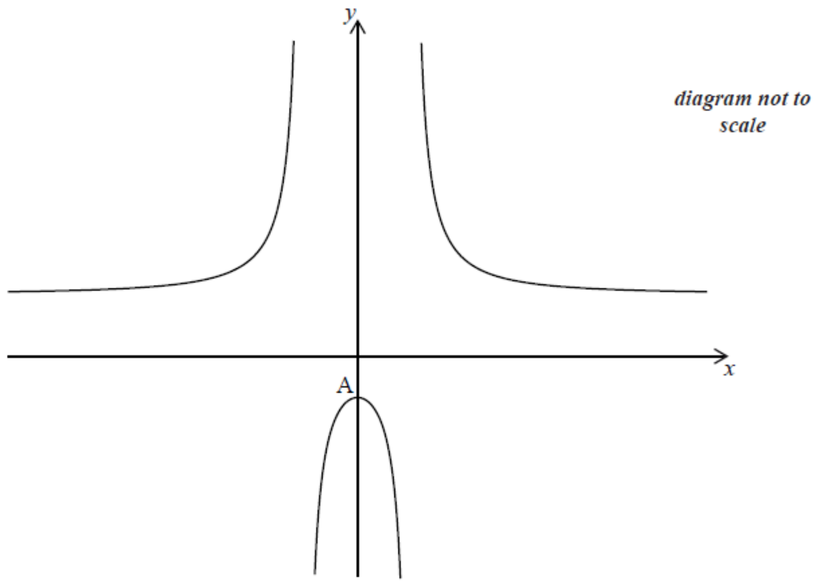
$p = -16$ ***A1*** ***N3***

[4 marks]

Let

$f(x) = 3 + \frac{20}{x^2-4}$, for

$x \neq \pm 2$. The graph of *f* is given below.



The *y*-intercept is at the point A.

10a. (i) Find the coordinates of A. *[7 marks]*

(ii) Show that $f'(x) = 0$ at A.

Markscheme

(i) coordinates of A are $(0,\ -2)$ ***A1A1*** ***N2***

(ii) derivative of $x^2 - 4 = 2x$ (seen anywhere) ***(A1)***

evidence of correct approach ***(M1)***

e.g. quotient rule, chain rule

finding $f'(x)$ ***A2***

e.g. $f'(x) = 20 \times (-1) \times (x^2 - 4)^{-2} \times (2x)$, $\frac{(x^2-4)(0)-(20)(2x)}{(x^2-4)^2}$

substituting $x = 0$ into $f'(x)$ (do not accept solving $f'(x) = 0$) ***M1***

at A $f'(x) = 0$ ***AG*** ***N0***

[7 marks]

10b. The second derivative $f''(x) = \frac{40(3x^2+4)}{(x^2-4)^3}$. Use this to *[6 marks]*

(i) justify that the graph of *f* has a local maximum at A;

(ii) explain why the graph of *f* does **not** have a point of inflexion.

Markscheme

(i) reference to $f'(x) = 0$ (seen anywhere) ***(R1)***

reference to $f''(0)$ is negative (seen anywhere) ***R1***

evidence of substituting $x = 0$ into $f''(x)$ ***M1***

finding $f''(0) = \frac{40\times4}{(-4)^3}$ $\left(= -\frac{5}{2}\right)$ ***A1***

then the graph must have a local maximum ***AG***

(ii) reference to $f''(x) = 0$ at point of inflexion ***(R1)***

recognizing that the second derivative is never 0 ***A1*** ***N2***

e.g. $40(3x^2 + 4) \neq 0$, $3x^2 + 4 \neq 0$, $x^2 \neq -\frac{4}{3}$, the numerator is always positive

Note: Do not accept the use of the first derivative in part (b).

[6 marks]

10c. Describe the behaviour of the graph of f for large $|x|$. *[1 mark]*

Markscheme

correct (informal) statement, including reference to approaching $y = 3$ ***A1*** ***N1***

e.g. getting closer to the line $y = 3$, horizontal asymptote at $y = 3$

[1 mark]

10d. Write down the range of f . *[2 marks]*

Markscheme

correct inequalities, $y \leq -2$, $y > 3$, ***FT*** from (a)(i) and (c) ***A1A1*** ***N2***

[2 marks]