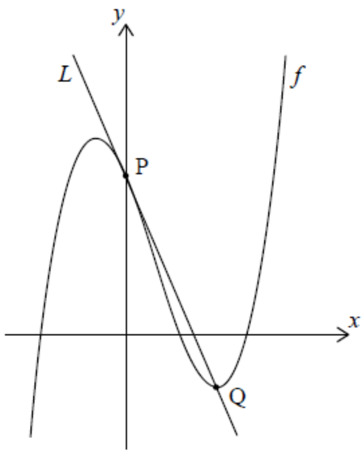
CalculusReview *[328 marks]*

Let $f(x) = x^3 - 2x^2 + ax + 6$. Part of the graph of f is shown in the following diagram.



The graph of f crosses the y-axis at the point P. The line L is tangent to the graph of f at P.

1a. Find the coordinates of P. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $f(0),\ 0^3 - 2(0)^2 + a(0) + 6,\ f(0) = 6,\ (0,\ y)$

(0, 6) (accept x = 0 **and** y = 6) ***A1 N2***

[2 marks]

1b. Find $f'(x)$. *[2 marks]*

Markscheme

$f' = 3x^2 - 4x + a$ ***A2 N2***

[2 marks]

1c. Hence, find the equation of L in terms of a. *[4 marks]*

Markscheme

valid approach ***(M1)***

eg $f'(0)$

correct working ***(A1)***

eg $3(0)^2 - 4(0) + a$, slope = a, $f'(0) = a$

attempt to substitute gradient and coordinates into linear equation ***(M1)***

eg $y - 6 = a(x - 0)$, $y - 0 = a(x - 6)$, $6 = a(0) + c$, $L = ax + 6$

correct equation ***A1 N3***

eg $y = ax + 6$, $y - 6 = ax$, $y - 6 = a(x - 0)$

[4 marks]

1d. The graph of f has a local minimum at the point Q. The line L passes through Q. *[8 marks]*

Find the value of a.

Markscheme

valid approach to find intersection ***(M1)***

eg $f(x) = L$

correct equation ***(A1)***

eg $x^3 - 2x^2 + ax + 6 = ax + 6$

correct working ***(A1)***

eg $x^3 - 2x^2 = 0$, $x^2(x - 2) = 0$

$x = 2$ at Q ***(A1)***

valid approach to find minimum ***(M1)***

eg $f'(x) = 0$

correct equation ***(A1)***

eg $3x^2 - 4x + a = 0$

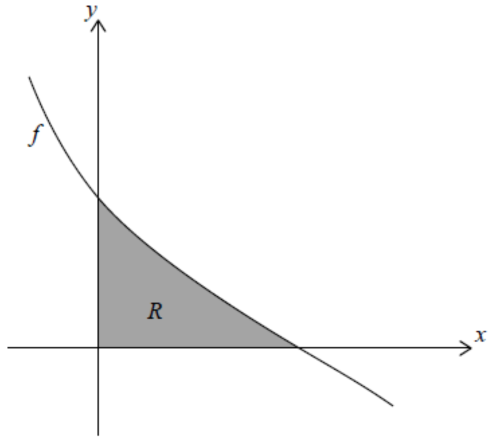
substitution of **their** value of x at Q into **their** $f'(x) = 0$ equation ***(M1)***

eg $3(2)^2 - 4(2) + a = 0$, $12 - 8 + a = 0$

$a = -4$ ***A1 N0***

[8 marks]

2. Let $f(x)=\frac{6-2x}{\sqrt{16+6x-x^2}}$. The following diagram shows part of the graph of f. *[8 marks]*



The region *R* is enclosed by the graph of f, the x-axis, and the y-axis. Find the area of *R*.

Markscheme

METHOD 1 (limits in terms of x)

valid approach to find x-intercept ***(M1)***

eg $f(x)=0,\ \frac{6-2x}{\sqrt{16+6x-x^2}}=0,\ 6-2x=0$

x-intercept is 3 ***(A1)***

valid approach using substitution or inspection ***(M1)***

eg $u=16+6x-x^2,\ \int_0^3 \frac{6-2x}{\sqrt{u}}\mathrm{d}x,\ \mathrm{d}u=6-2x,\ \int\frac{1}{\sqrt{u}},\ 2u^{\frac{1}{2}},$

$u=\sqrt{16+6x-x^2},\ \frac{\mathrm{d}u}{\mathrm{d}x}=(6-2x)\frac{1}{2}\left(16+6x-x^2\right)^{-\frac{1}{2}},\ \int 2\,\mathrm{d}u,\ 2u$

$\int f(x)\,\mathrm{d}x=2\sqrt{16+6x-x^2}$ ***(A2)***

substituting **both** of **their** limits into **their** integrated function and subtracting ***(M1)***

eg $2\sqrt{16+6(3)-3^2}-2\sqrt{16+6(0)^2-0^2},\ 2\sqrt{16+18-9}-2\sqrt{16}$

Note: Award ***M0*** if they substitute into original or differentiated function. Do not accept only "– 0" as evidence of substituting lower limit.

correct working ***(A1)***

eg $2\sqrt{25}-2\sqrt{16},\ 10-8$

area = 2 ***A1 N2***

METHOD 2 (limits in terms of u)

valid approach to find x-intercept ***(M1)***

eg $f(x)=0,\ \frac{6-2x}{\sqrt{16+6x-x^2}}=0,\ 6-2x=0$

x-intercept is 3 ***(A1)***

valid approach using substitution or inspection ***(M1)*** *eg* $u = 16 + 6x - x^2,\ \int_0^3 \frac{6-2x}{\sqrt{u}}\mathrm{d}x,\ \mathrm{d}u = 6 - 2x,\ \int \frac{1}{\sqrt{u}},$

$u = \sqrt{16 + 6x - x^2},\quad \frac{\mathrm{d}u}{\mathrm{d}x} = (6 - 2x)\frac{1}{2}\left(16 + 6x - x^2\right)^{-\frac{1}{2}},\ \int 2\,\mathrm{d}u$

correct integration ***(A2)***

eg $\int \frac{1}{\sqrt{u}}\mathrm{d}u = 2u^{\frac{1}{2}},\quad \int 2\,\mathrm{d}u = 2u$

both correct limits for u ***(A1)***

eg u = 16 and u = 25, $\int_{16}^{25} \frac{1}{\sqrt{u}}\mathrm{d}u,\quad \left[2u^{\frac{1}{2}}\right]_{16}^{25},$ u = 4 and u = 5, $\int_4^5 2\,\mathrm{d}u,\quad [2u]_4^5$

substituting **both** of **their** limits for u (do not accept 0 and 3) into **their** integrated function and subtracting ***(M1)***

eg $2\sqrt{25} - 2\sqrt{16},\ 10 - 8$

Note: Award ***M0*** if they substitute into original or differentiated function, or if they have not attempted to find limits for u.

area = 2 ***A1 N2***

[8 marks]

Let $f(x) = \frac{1}{\sqrt{2x-1}}$, for $x > \frac{1}{2}$.

3a. Find $\int (f(x))^2 \mathrm{d}x$. *[3 marks]*

Markscheme

correct working ***(A1)***

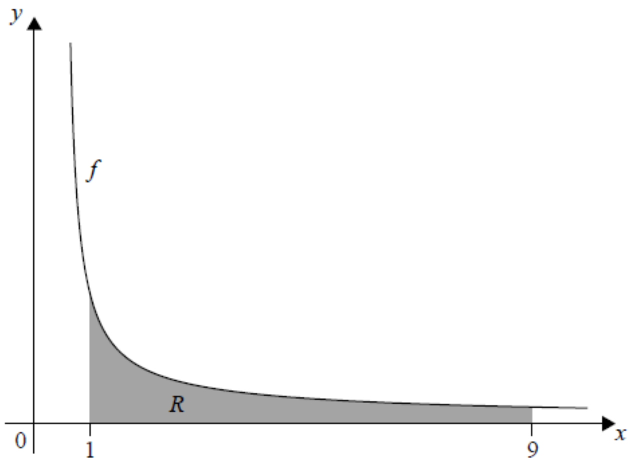
eg $\int \frac{1}{2x-1}\mathrm{d}x,\ \int (2x-1)^{-1},\ \frac{1}{2x-1},\ \int \left(\frac{1}{\sqrt{u}}\right)^2 \frac{\mathrm{d}u}{2}$

$\int (f(x))^2 \mathrm{d}x = \frac{1}{2}\ln(2x-1) + c$ ***A2 N3***

Note: Award ***A1*** for $\frac{1}{2}\ln(2x-1)$.

[3 marks]

3b. Part of the graph of f is shown in the following diagram. *[4 marks]*



The shaded region R is enclosed by the graph of f, the x-axis, and the lines $x = 1$ and $x = 9$. Find the volume of the solid formed when R is revolved 360° about the x-axis.

Markscheme

attempt to substitute either limits or the function into formula involving f^2 (accept absence of π / dx) ***(M1)***

eg $\int_1^9 y^2 \mathrm{d}x,\ \pi\int\left(\frac{1}{\sqrt{2x-1}}\right)^2 \mathrm{d}x,\ \left[\frac{1}{2}\ln(2x-1)\right]_1^9$

substituting limits into **their** integral and subtracting (in any order) ***(M1)***

eg $\frac{\pi}{2}(\ln(17) - \ln(1)),\ \pi\left(0 - \frac{1}{2}\ln(2 \times 9 - 1)\right)$

correct working involving calculating a log value or using log law ***(A1)***

eg $\ln(1) = 0,\ \ln\left(\frac{17}{1}\right)$

$\frac{\pi}{2}\ln 17\quad\left(\text{accept } \pi\ln\sqrt{17}\right)$ ***A1 N3***

Note: Full ***FT*** may be awarded as normal, from their incorrect answer in part (a), however, do not award the final two ***A*** marks unless they involve logarithms.

[4 marks]

4. Consider $f(x)$, $g(x)$ and $h(x)$, for $x \in \mathbb{R}$ where $h(x) = (f \circ g)(x)$. *[7 marks]*

Given that $g(3) = 7$, $g'(3) = 4$ and $f'(7) = -5$, find the gradient of the normal to the curve of h at $x = 3$.

Markscheme

recognizing the need to find h' ***(M1)***

recognizing the need to find $h'(3)$ (seen anywhere) ***(M1)***

evidence of choosing chain rule ***(M1)***

eg $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, $f'(g(3)) \times g'(3)$, $f'(g) \times g'$

correct working ***(A1)***

eg $f'(7) \times 4$, -5×4

$h'(3) = -20$ ***(A1)***

evidence of taking **their** negative reciprocal for normal ***(M1)***

eg $-\frac{1}{h'(3)}$, $m_1 m_2 = -1$

gradient of normal is $\frac{1}{20}$ ***A1 N4***

[7 marks]

A function $f(x)$ has derivative $f'(x) = 3x^2 + 18x$. The graph of f has an x-intercept at $x = -1$.

5a. Find $f(x)$. *[6 marks]*

Markscheme

evidence of integration ***(M1)***

eg $\int f'(x)$

correct integration (accept absence of C) ***(A1)(A1)***

eg $x^3 + \frac{18}{2}x^2 + C$, $x^3 + 9x^2$

attempt to substitute $x = -1$ into **their** $f = 0$ (must have C) ***M1***

eg $(-1)^3 + 9(-1)^2 + C = 0$, $-1 + 9 + C = 0$

Note: Award ***M0*** if they substitute into original or differentiated function.

correct working ***(A1)***

eg $8 + C = 0$, $C = -8$

$f(x) = x^3 + 9x^2 - 8$ ***A1 N5***

[6 marks]

5b. The graph of f has a point of inflexion at $x = p$. Find p. *[4 marks]*

Markscheme

METHOD 1 (using 2nd derivative)

recognizing that $f'' = 0$ (seen anywhere) ***M1***

correct expression for f'' ***(A1)***

eg $6x + 18$, $6p + 18$

correct working ***(A1)***

$6p + 18 = 0$

$p = -3$ ***A1 N3***

METHOD 1 (using 1st derivative)

recognizing the vertex of f' is needed ***(M2)***

eg $-\frac{b}{2a}$ (must be clear this is for f')

correct substitution ***(A1)***

eg $\frac{-18}{2\times3}$

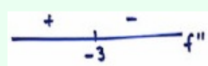
$p = -3$ ***A1 N3***

[4 marks]

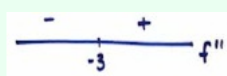
5c. Find the values of x for which the graph of f is concave-down. *[3 marks]*

Markscheme

valid attempt to use $f''(x)$ to determine concavity ***(M1)***

eg $f''(x) < 0$, $f''(-2)$, $f''(-4)$, $6x + 18 \le 0$ 

correct working ***(A1)***

eg $6x + 18 < 0$, $f''(-2) = 6$, $f''(-4) = -6$ 

f concave down for $x < -3$ (do not accept $x \le -3$) ***A1 N2***

[3 marks]

Consider a function f. The line L_1 with equation $y = 3x + 1$ is a tangent to the graph of f when $x = 2$

6a. Write down $f'(2)$. *[2 marks]*

Markscheme

recognize that $f'(x)$ is the gradient of the tangent at x ***(M1)***

eg $f'(x) = m$

$f'(2) = 3$ (accept *m* = 3) ***A1 N2***

[2 marks]

6b. Find $f(2)$. *[2 marks]*

Markscheme

recognize that $f(2) = y(2)$ ***(M1)***

eg $f(2) = 3 \times 2 + 1$

$f(2) = 7$ ***A1 N2***

[2 marks]

Let $g(x) = f(x^2 + 1)$ and P be the point on the graph of g where $x = 1$.

6c. Show that the graph of g has a gradient of 6 at P. *[5 marks]*

Markscheme

recognize that the gradient of the graph of g is $g'(x)$ ***(M1)***

choosing chain rule to find $g'(x)$ ***(M1)***

eg $\frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$, $u = x^2 + 1,\ u' = 2x$

$g'(x) = f'(x^2 + 1) \times 2x$ ***A2***

$g'(1) = 3 \times 2$ ***A1***

$g'(1) = 6$ ***AG N0***

[5 marks]

6d. Let L_2 be the tangent to the graph of g at P. L_1 intersects L_2 at the point Q. *[7 marks]*

Find the y-coordinate of Q.

Markscheme

at Q, $L_1 = L_2$ (seen anywhere) ***(M1)***

recognize that the gradient of L_2 is $g'(1)$ (seen anywhere) ***(M1)***
eg $m = 6$

finding $g(1)$ (seen anywhere) ***(A1)***
eg $g(1) = f(2)$, $g(1) = 7$

attempt to substitute gradient and/or coordinates into equation of a straight line ***M1***
eg $y - g(1) = 6(x - 1)$, $y - 1 = g'(1)(x - 7)$, $7 = 6(1) + \mathrm{b}$

correct equation for L_2

eg $y - 7 = 6(x - 1)$, $y = 6x + 1$ ***A1***

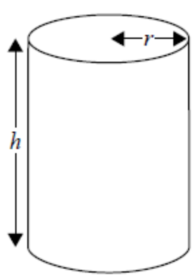
correct working to find Q ***(A1)***
eg same *y*-intercept, $3x = 0$

$y = 1$ ***A1 N2***

[7 marks]

A closed cylindrical can with radius r centimetres and height h centimetres has a volume of 20 π cm^3.

diagram not to scale



7a. Express *h* in terms of *r*. *[2 marks]*

Markscheme

correct equation for volume ***(A1)***
eg $\pi r^2 h = 20\pi$

$h = \frac{20}{r^2}$ ***A1 N2***

[2 marks]

The material for the base and top of the can costs 10 cents per cm ^{2} and the material for the curved side costs 8 cents per cm^2. The total cost of the material, in cents, is C.

7b. Show that $C = 20\pi r^2 + \frac{320\pi}{r}$. *[4 marks]*

Markscheme

attempt to find formula for cost of parts ***(M1)***
eg 10 × two circles, 8 × curved side

correct expression for cost of two circles in terms of *r* (seen anywhere) ***A1***
eg $2\pi r^2 \times 10$

correct expression for cost of curved side (seen anywhere) ***(A1)***
eg $2\pi r \times h \times 8$

correct expression for cost of curved side in terms of *r* ***A1***
eg $8 \times 2\pi r \times \frac{20}{r^2}$, $\frac{320\pi}{r^2}$

$C = 20\pi r^2 + \frac{320\pi}{r}$ ***AG N0***

[4 marks]

7c. Given that there is a minimum value for C, find this minimum value in terms of π. *[9 marks]*

Markscheme

recognize $C' = 0$ at minimum ***(R1)***
eg $C' = 0$, $\frac{\mathrm{d}C}{\mathrm{d}r} = 0$

correct differentiation (may be seen in equation)

$C' = 40\pi r - \frac{320\pi}{r^2}$ ***A1A1***

correct equation ***A1***
eg $40\pi r - \frac{320\pi}{r^2} = 0$, $40\pi r \frac{320\pi}{r^2}$

correct working ***(A1)***
eg $40r^3 = 320$, $r^3 = 8$

r = 2 (m) ***A1***

attempt to substitute **their** value of *r* into *C*
eg $20\pi \times 4 + 320 \times \frac{\pi}{2}$ ***(M1)***

correct working
eg $80\pi + 160\pi$ ***(A1)***

240π (cents) ***A1 N3***

Note: Do not accept 753.6, 753.98 or 754, even if 240 π is seen.

[9 marks]

Let $f(x) = 1 + e^{-x}$ and $g(x) = 2x + b$, for $x \in \mathbb{R}$, where b is a constant.

8a. Find $(g \circ f)(x)$. *[2 marks]*

Markscheme

attempt to form composite *(M1)*

eg $g(1 + e^{-x})$

correct function ***A1 N2***

eg $(g \circ f)(x) = 2 + b + 2e^{-x},\ 2(1 + e^{-x}) + b$

[2 marks]

8b. Given that $\lim_{x \to +\infty} (g \circ f)(x) = -3$, find the value of b. *[4 marks]*

Markscheme

evidence of $\lim_{x \to \infty} (2 + b + 2e^{-x}) = 2 + b + \lim_{x \to \infty} (2e^{-x})$ ***(M1)***

eg $2 + b + 2e^{-\infty}$, graph with horizontal asymptote when $x \to \infty$

Note: Award ***M0*** if candidate clearly has incorrect limit, such as $x \to 0,\ e^{\infty},\ 2e^{0}$.

evidence that $e^{-x} \to 0$ (seen anywhere) ***(A1)***

eg $\lim_{x \to \infty} (e^{-x}) = 0,\ 1 + e^{-x} \to 1,\ 2(1) + b = -3,\ e^{\text{large negative number}} \to 0$, graph of $y = e^{-x}$ or

$y = 2e^{-x}$ with asymptote $y = 0$, graph of composite function with asymptote $y = -3$

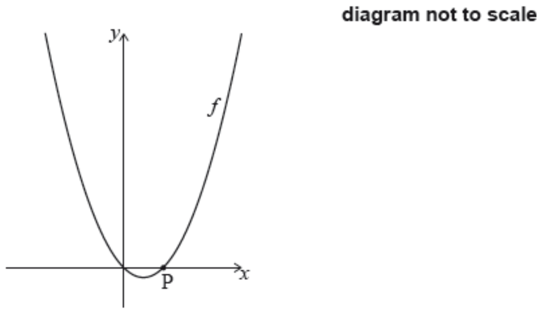
correct working ***(A1)***

eg $2 + b = -3$

$b = -5$ ***A1 N2***

[4 marks]

Let $f(x) = x^2 - x$, for $x \in \mathbb{R}$. The following diagram shows part of the graph of f.



The graph of f crosses the x-axis at the origin and at the point $\mathrm{P}(1,\ 0)$.

9a. Show that $f'(1) = 1$. *[3 marks]*

Markscheme

$f'(x) = 2x - 1$ ***A1A1***

correct substitution ***A1***

eg $2(1) - 1,\ 2 - 1$

$f'(1) = 1$ ***AG*** ***N0***

[3 marks]

The line *L* is the normal to the graph of *f* at P.

9b. Find the equation of L in the form $y = ax + b$. *[3 marks]*

Markscheme

correct approach to find the gradient of the normal ***(A1)***

eg $\frac{-1}{f'(1)},\ m_1 m_2 = -1,\ \text{slope} = -1$

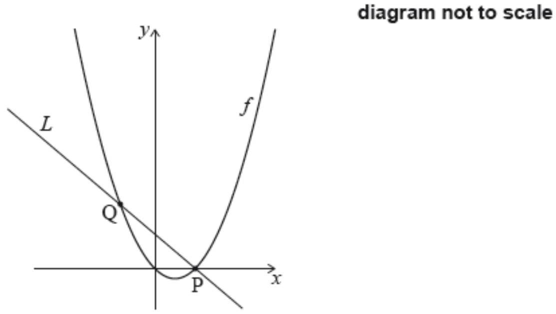
attempt to substitute correct normal gradient and coordinates into equation of a line ***(M1)***

eg $y - 0 = -1(x - 1),\ 0 = -1 + b,\ b = 1,\ L = -x + 1$

$y = -x + 1$ ***A1*** ***N2***

[3 marks]

The line L intersects the graph of f at another point Q, as shown in the following diagram.



9c. Find the x-coordinate of Q. *[4 marks]*

Markscheme

equating expressions ***(M1)***

eg $f(x) = L,\ -x + 1 = x^2 - x$

correct working (must involve combining terms) ***(A1)***

eg $x^2 - 1 = 0,\ x^2 = 1,\ x = 1$

$x = -1 \quad (\text{accept } Q(-1,\ 2))$ ***A2 N3***

[4 marks]

9d. Find the area of the region enclosed by the graph of f and the line L. *[6 marks]*

Markscheme

valid approach ***(M1)***

eg $\int L - f,\ \int_{-1}^{1} (1 - x^2)\mathrm{d}x$, splitting area into triangles and integrals

correct integration ***(A1)(A1)***

eg $\left[x - \frac{x^3}{3}\right]_{-1}^{1},\ -\frac{x^3}{3} - \frac{x^2}{2} + \frac{x^2}{2} + x$

substituting **their** limits into **their** integrated function and subtracting (in any order) ***(M1)***

eg $1 - \frac{1}{3} - \left(-1 - \frac{-1}{3}\right)$

Note: Award ***M0*** for substituting into original or differentiated function.

area $= \frac{4}{3}$ ***A2 N3***

[6 marks]

10a. Find $\int x\mathrm{e}^{x^2-1}\mathrm{d}x$. *[4 marks]*

Markscheme

valid approach to set up integration by substitution/inspection ***(M1)***

eg $u = x^2 - 1,\ \mathrm{d}u = 2x,\ \int 2x\mathrm{e}^{x^2-1}\mathrm{d}x$

correct expression ***(A1)***

eg $\frac{1}{2}\int 2x\mathrm{e}^{x^2-1}\mathrm{d}x,\ \frac{1}{2}\int \mathrm{e}^{u}\mathrm{d}u$

$\frac{1}{2}\mathrm{e}^{x^2-1} + c$ ***A2*** ***N4***

Notes: Award ***A1*** if missing "$+c$".

[4 marks]

10b. Find $f(x)$, given that $f'(x) = x\mathrm{e}^{x^2-1}$ and $f(-1) = 3$. *[3 marks]*

Markscheme

substituting $x = -1$ into **their** answer from (a) ***(M1)***

eg $\frac{1}{2}\mathrm{e}^{0},\ \frac{1}{2}\mathrm{e}^{1-1} = 3$

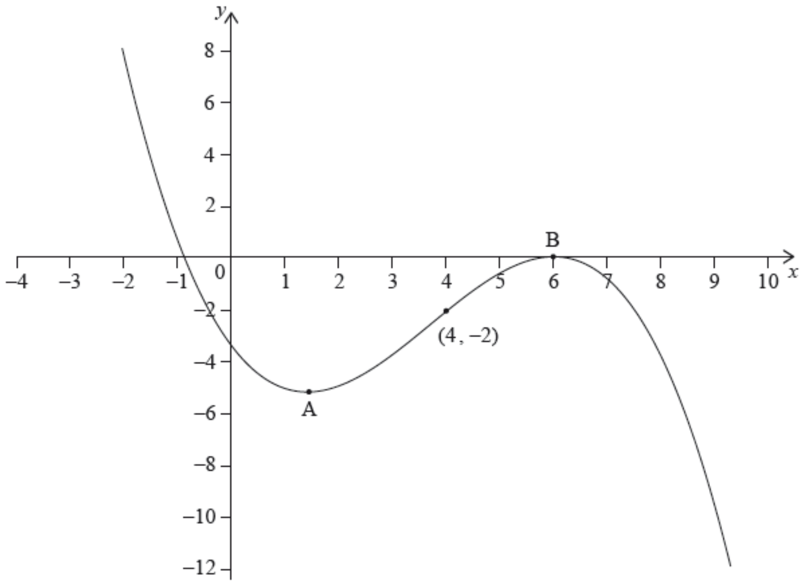
correct working ***(A1)***

eg $\frac{1}{2} + c = 3,\ c = 2.5$

$f(x) = \frac{1}{2}\mathrm{e}^{x^2-1} + 2.5$ ***A1*** ***N2***

[3 marks]

The following diagram shows the graph of f', the derivative of f.



The graph of f' has a local minimum at A, a local maximum at B and passes through $(4,\ -2)$.

The point $\mathrm{P}(4,\ 3)$ lies on the graph of the function, f.

11a. Write down the gradient of the curve of f at P. *[1 mark]*

Markscheme

-2 ***A1*** ***N1***

[1 mark]

11b. Find the equation of the normal to the curve of f at P. *[3 marks]*

Markscheme

gradient of normal $= \frac{1}{2}$ ***(A1)***

attempt to substitute their normal gradient and coordinates of P (in any order) ***(M1)***

eg $y - 4 = \frac{1}{2}(x-3),\ 3 = \frac{1}{2}(4) + b,\ b = 1$

$y - 3 = \frac{1}{2}(x-4),\ y = \frac{1}{2}x + 1,\ x - 2y + 2 = 0$ ***A1*** ***N3***

[3 marks]

11c. Determine the concavity of the graph of f when $4 < x < 5$ **and** justify your answer. *[2 marks]*

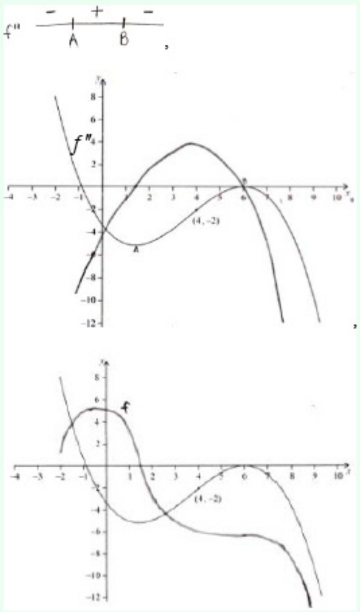
Markscheme

correct answer **and** valid reasoning ***A2*** ***N2***

answer: *eg* graph of f is concave up, concavity is positive (between $4 < x < 5$)

reason: *eg* slope of f' is positive, f' is increasing, $f'' > 0$,

sign chart (must clearly be for f'' and show A and B)



Note: The reason given must refer to a specific function/graph. Referring to "the graph" or "it" is not sufficient.

[2 marks]

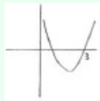
A quadratic function f can be written in the form $f(x) = a(x - p)(x - 3)$. The graph of f has axis of symmetry $x = 2.5$ and y-intercept at $(0, \ -6)$

12a. Find the value of p. *[3 marks]*

Markscheme

METHOD 1 (using *x*-intercept)

determining that 3 is an x-intercept ***(M1)***

eg $x - 3 = 0,$ 

valid approach ***(M1)***

eg $3 - 2.5,\ \frac{p+3}{2} = 2.5$

$p = 2$ ***A1 N2***

METHOD 2 (expanding *f* (*x*))

correct expansion (accept absence of a) ***(A1)***

eg $ax^2 - a(3+p)x + 3ap,\ x^2 - (3+p)x + 3p$

valid approach involving equation of axis of symmetry ***(M1)***

eg $\frac{-b}{2a} = 2.5,\ \frac{a(3+p)}{2a} = \frac{5}{2},\ \frac{3+p}{2} = \frac{5}{2}$

$p = 2$ ***A1 N2***

METHOD 3 (using derivative)

correct derivative (accept absence of a) ***(A1)***

eg $a(2x - 3 - p),\ 2x - 3 - p$

valid approach ***(M1)***

eg $f'(2.5) = 0$

$p = 2$ ***A1 N2***

[3 marks]

12b. Find the value of a. *[3 marks]*

Markscheme

attempt to substitute $(0,\ -6)$ ***(M1)***

eg $-6 = a(0-2)(0-3),\ 0 = a(-8)(-9),\ a(0)^2 - 5a(0) + 6a = -6$

correct working ***(A1)***

eg $-6 = 6a$

$a = -1$ ***A1 N2***

[3 marks]

12c. The line $y = kx - 5$ is a tangent to the curve of f. Find the values of k. *[8 marks]*

Markscheme

METHOD 1 (using discriminant)

recognizing tangent intersects curve once ***(M1)***

recognizing one solution when discriminant = 0 ***M1***

attempt to set up equation ***(M1)***

eg $g = f,\ kx - 5 = -x^2 + 5x - 6$

rearranging their equation to equal zero ***(M1)***

eg $x^2 - 5x + kx + 1 = 0$

correct discriminant (if seen explicitly, not just in quadratic formula) ***A1***

eg $(k-5)^2 - 4,\ 25 - 10k + k^2 - 4$

correct working ***(A1)***

eg $k - 5 = \pm 2,\ (k-3)(k-7) = 0,\ \frac{10 \pm \sqrt{100 - 4 \times 21}}{2}$

$k = 3,\ 7$ ***A1A1*** ***N0***

METHOD 2 (using derivatives)

attempt to set up equation ***(M1)***

eg $g = f,\ kx - 5 = -x^2 + 5x - 6$

recognizing derivative/slope are equal ***(M1)***

eg $f' = m_T,\ f' = k$

correct derivative of f ***(A1)***

eg $-2x + 5$

attempt to set up equation in terms of either x or k ***M1***

eg $(-2x+5)x - 5 = -x^2 + 5x - 6,\ k\left(\frac{5-k}{2}\right) - 5 = -\left(\frac{5-k}{2}\right)^2 + 5\left(\frac{5-k}{2}\right) - 6$

rearranging their equation to equal zero ***(M1)***

eg $x^2 - 1 = 0,\ k^2 - 10k + 21 = 0$

correct working ***(A1)***

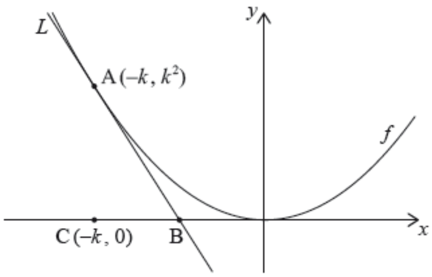
eg $x = \pm 1,\ (k-3)(k-7) = 0,\ \frac{10 \pm \sqrt{100 - 4 \times 21}}{2}$

$k = 3,\ 7$ ***A1A1*** ***N0***

[8 marks]

Let $f(x) = x^2$. The following diagram shows part of the graph of f.

diagram not to scale



The line L is the tangent to the graph of f at the point $\mathrm{A}(-k,\ k^2)$, and intersects the x-axis at point B. The point C is $(-k,\ 0)$.

13a. Write down $f'(x)$. *[1 mark]*

Markscheme

$f'(x) = 2x$ ***A1*** ***N1***

[1 mark]

13b. Find the gradient of L. *[2 marks]*

Markscheme

attempt to substitute $x = -k$ into their derivative ***(M1)***

gradient of L is $-2k$ ***A1*** ***N2***

[2 marks]

13c. Show that the x-coordinate of B is $-\frac{k}{2}$. *[5 marks]*

Markscheme

METHOD 1

attempt to substitute coordinates of A and their gradient into equation of a line ***(M1)***

eg $k^2 = -2k(-k) + b$

correct equation of L in any form ***(A1)***

eg $y - k^2 = -2k(x + k),\ y = -2kx - k^2$

valid approach ***(M1)***

eg $y = 0$

correct substitution into L equation ***A1***

eg $-k^2 = -2kx - 2k^2,\ 0 = -2kx - k^2$

correct working ***A1***

eg $2kx = -k^2$

$x = -\frac{k}{2}$ ***AG*** ***N0***

METHOD 2

valid approach ***(M1)***

eg $\text{gradient} = \frac{y_2-y_1}{x_2-x_1},\ -2k = \frac{\text{rise}}{\text{run}}$

recognizing $y = 0$ at B ***(A1)***

attempt to substitute coordinates of A and B into slope formula ***(M1)***

eg $\frac{k^2-0}{-k-x},\ \frac{-k^2}{x+k}$

correct equation ***A1***

eg $\frac{k^2-0}{-k-x} = -2k,\ \frac{-k^2}{x+k} = -2k,\ -k^2 = -2k(x + k)$

correct working ***A1***

eg $2kx = -k^2$

$x = -\frac{k}{2}$ ***AG*** ***N0***

[5 marks]

13d. Find the area of triangle ABC, giving your answer in terms of k. *[2 marks]*

Markscheme

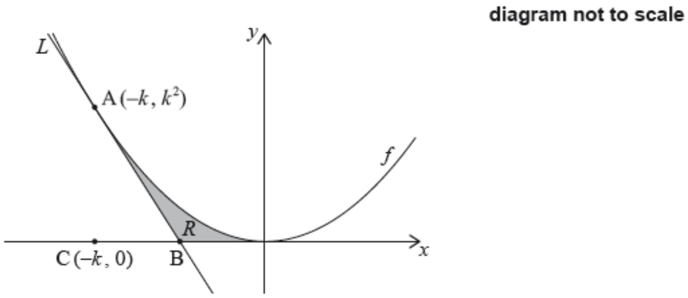
valid approach to find area of triangle ***(M1)***

eg $\frac{1}{2}(k^2)\left(\frac{k}{2}\right)$

area of $\text{ABC} = \frac{k^3}{4}$ ***A1*** ***N2***

[2 marks]

The region R is enclosed by L, the graph of f, and the x-axis. This is shown in the following diagram.



13e. Given that the area of triangle ABC is p times the area of R, find the value of p. *[7 marks]*

Markscheme

METHOD 1 ($\int f - \text{triangle}$)

valid approach to find area from $-k$ to 0 ***(M1)***

eg $\int_{-k}^{0} x^2 \mathrm{d}x,\ \int_{0}^{-k} f$

correct integration (seen anywhere, even if ***M0*** awarded) ***A1***

eg $\frac{x^3}{3},\ \left[\frac{1}{3}x^3\right]_{-k}^{0}$

substituting **their** limits into **their** integrated function and subtracting ***(M1)***

eg $0 - \frac{(-k)^3}{3}$, area from $-k$ to 0 is $\frac{k^3}{3}$

Note: Award ***M0*** for substituting into original or differentiated function.

attempt to find area of R ***(M1)***

eg $\int_{-k}^{0} f(x)\mathrm{d}x - \text{ triangle}$

correct working for R ***(A1)***

eg $\frac{k^3}{3} - \frac{k^3}{4},\ R = \frac{k^3}{12}$

correct substitution into $\text{triangle} = pR$ ***(A1)***

eg $\frac{k^3}{4} = p\left(\frac{k^3}{3} - \frac{k^3}{4}\right),\ \frac{k^3}{4} = p\left(\frac{k^3}{12}\right)$

$p = 3$ ***A1*** ***N2***

METHOD 2 ($\int (f - L)$)

valid approach to find area of R ***(M1)***

eg $\int_{-k}^{-\frac{k}{2}} x^2 - (-2kx - k^2)\mathrm{d}x + \int_{-\frac{k}{2}}^{0} x^2 \mathrm{d}x,\ \int_{-k}^{-\frac{k}{2}} (f - L) + \int_{-\frac{k}{2}}^{0} f$

correct integration (seen anywhere, even if ***M0*** awarded) ***A2***

eg $\frac{x^3}{3} + kx^2 + k^2x,\ \left[\frac{x^3}{3} + kx^2 + k^2x\right]_{-k}^{-\frac{k}{2}} + \left[\frac{x^3}{3}\right]_{-\frac{k}{2}}^{0}$

substituting **their** limits into **their** integrated function and subtracting ***(M1)***

eg

$$\left(\frac{\left(-\frac{k}{2}\right)^3}{3}+k\left(-\frac{k}{2}\right)^2+k^2\left(-\frac{k}{2}\right)\right)-\left(\frac{(-k)^3}{3}+k(-k)^2+k^2(-k)\right)+(0)-\left(\frac{\left(-\frac{k}{2}\right)^3}{3}\right)$$

Note: Award ***M0*** for substituting into original or differentiated function.

correct working for R ***(A1)***

eg $\frac{k^3}{24}+\frac{k^3}{24},\ -\frac{k^3}{24}+\frac{k^3}{4}-\frac{k^3}{2}+\frac{k^3}{3}-k^3+k^3+\frac{k^3}{24},\ R=\frac{k^3}{12}$

correct substitution into $\text{triangle}=pR$ ***(A1)***

eg $\frac{k^3}{4}=p\left(\frac{k^3}{24}+\frac{k^3}{24}\right),\ \frac{k^3}{4}=p\left(\frac{k^3}{12}\right)$

$p=3$ ***A1*** ***N2***

[7 marks]

14. Let $f'(x)=\frac{3x^2}{\left(x^3+1\right)^5}$. Given that $f(0)=1$, find $f(x)$. *[6 marks]*

Markscheme

valid approach ***(M1)***

eg $\int f'\mathrm{d}x,\ \int\frac{3x^2}{\left(x^3+1\right)^5}\mathrm{d}x$

correct integration by substitution/inspection ***A2***

eg $f(x)=-\frac{1}{4}(x^3+1)^{-4}+c,\ \frac{-1}{4\left(x^3+1\right)^4}$

correct substitution into **their** integrated function (must include c) ***M1***

eg $1=\frac{-1}{4\left(0^3+1\right)^4}+c,\ -\frac{1}{4}+c=1$

Note: Award ***M0*** if candidates substitute into f' or f''.

$c=\frac{5}{4}$ ***(A1)***

$f(x)=-\frac{1}{4}(x^3+1)^{-4}+\frac{5}{4}\ \left(=\frac{-1}{4\left(x^3+1\right)^4}+\frac{5}{4},\ \frac{5\left(x^3+1\right)^4-1}{4\left(x^3+1\right)^4}\right)$ ***A1*** ***N4***

[6 marks]

Let $f(x)=\cos x$.

15a. (i) Find the first four derivatives of $f(x)$. *[4 marks]*

(ii) Find $f^{(19)}(x)$.

Markscheme

(i)

$f'(x) = -\sin x,\ f''(x) = -\cos x,\ f^{(3)}(x) = \sin x,\ f^{(4)}(x) = \cos x$ ***A2*** ***N2***

(ii) valid approach ***(M1)***

eg recognizing that 19 is one less than a multiple of 4, $f^{(19)}(x) = f^{(3)}(x)$

$f^{(19)}(x) = \sin x$ ***A1*** ***N2***

[4 marks]

Let $g(x) = x^k$, where $k \in \mathbb{Z}^+$.

15b. (i) Find the first three derivatives of $g(x)$. *[5 marks]*

(ii) Given that $g^{(19)}(x) = \frac{k!}{(k-p)!}(x^{k-19})$, find p.

Markscheme

(i)

$g'(x) = kx^{k-1}$

$g''(x) = k(k-1)x^{k-2},\ g^{(3)}(x) = k(k-1)(k-2)x^{k-3}$ ***A1A1*** ***N2***

(ii) **METHOD 1**

correct working that leads to the correct answer, involving the correct expression for the 19th derivative ***A2***

eg $k(k-1)(k-2)\ldots(k-18) \times \frac{(k-19)!}{(k-19)!},\ {}_kP_{19}$

$p = 19$ (accept $\frac{k!}{(k-19)!}x^{k-19}$) ***A1*** ***N1***

METHOD 2

correct working involving recognizing patterns in coefficients of first three derivatives (may be seen in part (b)(i)) leading to a general rule for 19th coefficient ***A2***

eg $g'' = 2!\binom{k}{2},\ k(k-1)(k-2) = \frac{k!}{(k-3)!},\ g^{(3)}(x) = {}_kP_3(x^{k-3})$

$g^{(19)}(x) = 19!\binom{k}{19},\ 19! \times \frac{k!}{(k-19)! \times 19!},\ {}_kP_{19}$

$p = 19$ (accept $\frac{k!}{(k-19)!}x^{k-19}$) ***A1*** ***N1***

[5 marks]

Let $k = 21$ and $h(x) = \left(f^{(19)}(x) \times g^{(19)}(x)\right)$.

15c. (i) Find $h'(x)$. *[7 marks]*

(ii) Hence, show that $h'(\pi) = \frac{-21!}{2}\pi^2$.

Markscheme

(i) valid approach using product rule ***(M1)***

eg $uv' + vu',\ f^{(19)}g^{(20)} + f^{(20)}g^{(19)}$

correct 20th derivatives (must be seen in product rule) ***(A1)(A1)***

eg $g^{(20)}(x) = \frac{21!}{(21-20)!}x,\ f^{(20)}(x) = \cos x$

$h'(x) = \sin x(21!x) + \cos x\left(\frac{21!}{2}x^2\right)\ \left(\text{accept } \sin x\left(\frac{21!}{1!}x\right) + \cos x\left(\frac{21!}{2!}x^2\right)\right)$ ***A1 N3***

(ii) substituting $x = \pi$ (seen anywhere) ***(A1)***

eg $f^{(19)}(\pi)g^{(20)}(\pi) + f^{(20)}(\pi)g^{(19)}(\pi),\ \sin\pi\frac{21!}{1!}\pi + \cos\pi\frac{21!}{2!}\pi^2$

evidence of one correct value for $\sin\pi$ or $\cos\pi$ (seen anywhere) ***(A1)***

eg $\sin\pi = 0,\ \cos\pi = -1$

evidence of correct values substituted into $h'(\pi)$ ***A1***

eg $21!(\pi)\left(0 - \frac{\pi}{2!}\right),\ 21!(\pi)\left(-\frac{\pi}{2}\right),\ 0 + (-1)\frac{21!}{2}\pi^2$

Note: If candidates write only the first line followed by the answer, award ***A1A0A0***.

$\frac{-21!}{2}\pi^2$ ***AG N0***

[7 marks]

Let $f(x) = \sqrt{4x+5}$, for $x \geqslant -1.25$.

16a. Find $f'(1)$. *[4 marks]*

Markscheme

choosing chain rule ***(M1)***

eg $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x},\ u = 4x + 5,\ u' = 4$

correct derivative of f ***A2***

eg $\frac{1}{2}(4x+5)^{-\frac{1}{2}} \times 4,\ f'(x) = \frac{2}{\sqrt{4x+5}}$

$f'(1) = \frac{2}{3}$ ***A1*** ***N2***

[4 marks]

Consider another function g. Let R be a point on the graph of g. The x-coordinate of R is 1. The equation of the tangent to the graph at R is $y = 3x + 6$.

16b. Write down $g'(1)$. *[2 marks]*

Markscheme

recognize that $g'(x)$ is the gradient of the tangent ***(M1)***

eg $g'(x) = m$

$g'(1) = 3$ ***A1*** ***N2***

[2 marks]

16c. Find $g(1)$. *[2 marks]*

Markscheme

recognize that R is on the tangent ***(M1)***

eg $g(1) = 3 \times 1 + 6$, sketch

$g(1) = 9$ ***A1*** ***N2***

[2 marks]

16d. Let $h(x) = f(x) \times g(x)$. Find the equation of the tangent to the graph of h at the point where $x = 1$. *[7 marks]*

Markscheme

$f(1) = \sqrt{4+5}\ (= 3)$ (seen anywhere) ***A1***

$h(1) = 3 \times 9\ (= 27)$ (seen anywhere) ***A1***

choosing product rule to find $h'(x)$ ***(M1)***

eg $uv' + u'v$

correct substitution to find $h'(1)$ ***(A1)***

eg $f(1) \times g'(1) + f'(1) \times g(1)$

$h'(1) = 3 \times 3 + \frac{2}{3} \times 9\ (= 15)$ ***A1***

EITHER

attempt to substitute coordinates (in any order) into the equation of a straight line ***(M1)***

eg $y - 27 = h'(1)(x - 1),\ y - 1 = 15(x - 27)$

$y - 27 = 15(x - 1)$ ***A1*** ***N2***

OR

attempt to substitute coordinates (in any order) to find the y-intercept ***(M1)***

eg $27 = 15 \times 1 + b,\ 1 = 15 \times 27 + b$

$y = 15x + 12$ ***A1*** ***N2***

[7 marks]

Let $f'(x) = \frac{6-2x}{6x-x^2}$, for $0 < x < 6$.

The graph of f has a maximum point at P.

17a. Find the x-coordinate of P. *[3 marks]*

Markscheme

recognizing $f'(x) = 0$ ***(M1)***

correct working ***(A1)***

eg $6 - 2x = 0$

$x = 3$ ***A1*** ***N2***

[3 marks]

The y-coordinate of P is $\ln 27$.

17b. Find $f(x)$, expressing your answer as a single logarithm. *[8 marks]*

Markscheme

evidence of integration ***(M1)***

eg $\int f',\ \int \frac{6-2x}{6x-x^2}\mathrm{d}x$

using substitution ***(A1)***

eg $\int \frac{1}{u}\mathrm{d}u$ where $u = 6x - x^2$

correct integral ***A1***

eg $\ln(u) + c,\ \ln(6x - x^2)$

substituting $(3,\ \ln 27)$ into **their** integrated expression (must have c) ***(M1)***

eg $\ln(6 \times 3 - 3^2) + c = \ln 27,\ \ln(18 - 9) + \ln k = \ln 27$

correct working ***(A1)***

eg $c = \ln 27 - \ln 9$

EITHER

$c = \ln 3$ ***(A1)***

attempt to substitute **their** value of c into $f(x)$ ***(M1)***

eg $f(x) = \ln(6x - x^2) + \ln 3$ ***A1 N4***

OR

attempt to substitute **their** value of c into $f(x)$ ***(M1)***

eg $f(x) = \ln(6x - x^2) + \ln 27 - \ln 9$

correct use of a log law ***(A1)***

eg $f(x) = \ln(6x - x^2) + \ln\left(\frac{27}{9}\right),\ f(x) = \ln\left(27(6x - x^2)\right) - \ln 9$

$f(x) = \ln\left(3(6x - x^2)\right)$ ***A1 N4***

[8 marks]

17c. The graph of f is transformed by a vertical stretch with scale factor $\frac{1}{\ln 3}$. The image of P under this transformation has coordinates $(a,\ b)$.

Find the value of a and of b, where $a,\ b \in \mathbb{N}$.

Markscheme

$a = 3$ ***A1*** ***N1***

correct working ***A1***

eg $\frac{\ln 27}{\ln 3}$

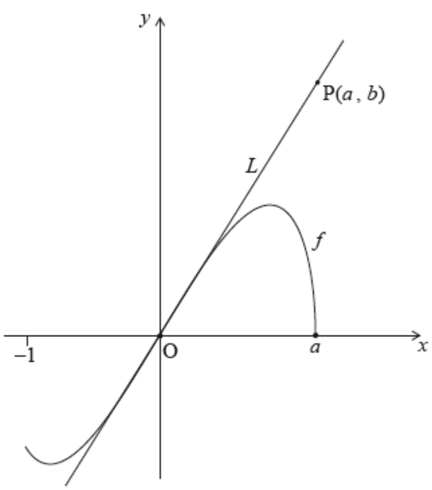
correct use of log law ***(A1)***

eg $\frac{3\ln 3}{\ln 3}$, $\log_3 27$

$b = 3$ ***A1*** ***N2***

[4 marks]

The following diagram shows the graph of $f(x) = 2x\sqrt{a^2 - x^2}$, for $-1 \leqslant x \leqslant a$, where $a > 1$
.



The line L is the tangent to the graph of f at the origin, O. The point $\mathrm{P}(a,\ b)$ lies on L.

18a. (i) Given that $f'(x) = \frac{2a^2 - 4x^2}{\sqrt{a^2 - x^2}}$, for $-1 \leqslant x < a$, find the equation of L. *[6 marks]*

(ii) Hence or otherwise, find an expression for b in terms of a.

Markscheme

(i) recognizing the need to find the gradient when $x = 0$ (seen anywhere) ***R1***

eg $f'(0)$

correct substitution ***(A1)***

$f'(0) = \frac{2a^2 - 4(0)}{\sqrt{a^2 - 0}}$

$f'(0) = 2a$ ***(A1)***

correct equation with gradient $2a$ (do not accept equations of the form $L = 2ax$) ***A1 N3***

eg $y = 2ax,\ y - b = 2a(x - a),\ y = 2ax - 2a^2 + b$

(ii) **METHOD 1**

attempt to substitute $x = a$ into **their** equation of L ***(M1)***

eg $y = 2a \times a$

$b = 2a^2$ ***A1 N2***

METHOD 2

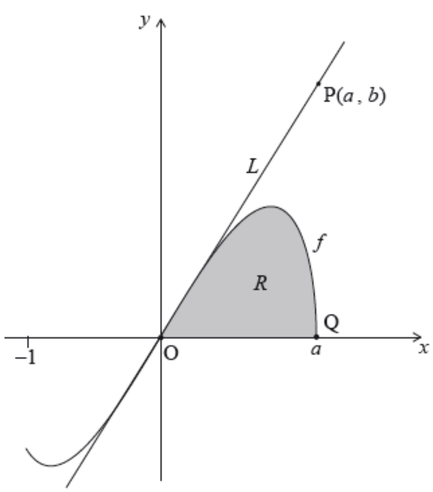
equating gradients ***(M1)***

eg $\frac{b}{a} = 2a$

$b = 2a^2$ ***A1 N2***

[6 marks]

The point $\mathrm{Q}(a,\ 0)$ lies on the graph of f. Let R be the region enclosed by the graph of f and the x-axis. This information is shown in the following diagram.



Let A_R be the area of the region R.

18b. Show that $A_R = \frac{2}{3}a^3$. *[6 marks]*

Markscheme

METHOD 1

recognizing that area $= \int_0^a f(x)\mathrm{d}x$ (seen anywhere) ***R1***

valid approach using substitution or inspection ***(M1)***

eg $\int 2x\sqrt{u}\mathrm{d}x,\ u = a^2 - x^2,\ \mathrm{d}u = -2x\mathrm{d}x,\ \frac{2}{3}(a^2 - x^2)^{\frac{3}{2}}$

correct working ***(A1)***

eg $\int 2x\sqrt{a^2 - x^2}\mathrm{d}x = \int -\sqrt{u}\mathrm{d}u$

$\int -\sqrt{u}\mathrm{d}u = -\frac{u^{\frac{3}{2}}}{\frac{3}{2}}$ ***(A1)***

$\int f(x)\mathrm{d}x = -\frac{2}{3}(a^2 - x^2)^{\frac{3}{2}} + c$ ***(A1)***

substituting limits and subtracting ***A1***

eg $A_R = -\frac{2}{3}(a^2 - a^2)^{\frac{3}{2}} + \frac{2}{3}(a^2 - 0)^{\frac{3}{2}},\ \frac{2}{3}(a^2)^{\frac{3}{2}}$

$A_R = \frac{2}{3}a^3$ ***AG*** ***N0***

METHOD 2

recognizing that area $= \int_0^a f(x)\mathrm{d}x$ (seen anywhere) ***R1***

valid approach using substitution or inspection ***(M1)***

eg $\int 2x\sqrt{u}\mathrm{d}x,\ u = a^2 - x^2,\ \mathrm{d}u = -2x\mathrm{d}x,\ \frac{2}{3}(a^2 - x^2)^{\frac{3}{2}}$

correct working ***(A1)***

eg $\int 2x\sqrt{a^2 - x^2}\mathrm{d}x = \int -\sqrt{u}\mathrm{d}u$

$\int -\sqrt{u}\mathrm{d}u = -\frac{u^{\frac{3}{2}}}{\frac{3}{2}}$ ***(A1)***

new limits for *u* (even if integration is incorrect) ***(A1)***

eg $u = 0 \text{ and } u = a^2,\ \int_0^{a^2} u^{\frac{1}{2}}\mathrm{d}u,\ \left[-\frac{2}{3}u^{\frac{3}{2}}\right]_{a^2}^{0}$

substituting limits and subtracting ***A1***

eg $A_R = -\left(0 - \frac{2}{3}a^3\right),\ \frac{2}{3}(a^2)^{\frac{3}{2}}$

$A_R = \frac{2}{3}a^3$ ***AG*** ***N0***

[6 marks]

18c. Let A_T be the area of the triangle OPQ. Given that $A_T = kA_R$, find the value of k. *[4 marks]*

Markscheme

METHOD 1

valid approach to find area of triangle ***(M1)***

eg $\frac{1}{2}(\text{OQ})(\text{PQ}),\ \frac{1}{2}ab$

correct substitution into formula for A_T (seen anywhere) ***(A1)***

eg $A_T = \frac{1}{2} \times a \times 2a^2,\ a^3$

valid attempt to find k (must be in terms of a) ***(M1)***

eg $a^3 = k\frac{2}{3}a^3,\ k = \frac{a^3}{\frac{2}{3}a^3}$

$k = \frac{3}{2}$ ***A1 N2***

METHOD 2

valid approach to find area of triangle ***(M1)***

eg $\int_0^a (2ax)\mathrm{d}x$

correct working ***(A1)***

eg $[ax^2]_0^a,\ a^3$

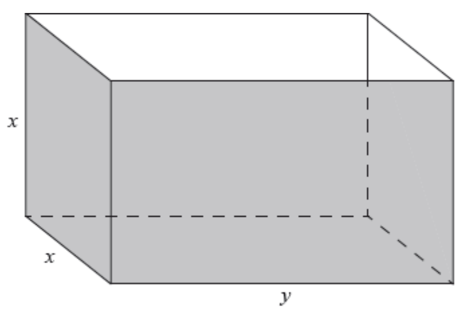
valid attempt to find k (must be in terms of a) ***(M1)***

eg $a^3 = k\frac{2}{3}a^3,\ k = \frac{a^3}{\frac{2}{3}a^3}$

$k = \frac{3}{2}$ ***A1 N2***

[4 marks]

Fred makes an open metal container in the shape of a cuboid, as shown in the following diagram.



The container has height x m, width x m and length y m. The volume is $36\ \text{m}^3$.

Let $A(x)$ be the outside surface area of the container.

19a. Show that $A(x) = \frac{108}{x} + 2x^2$. *[4 marks]*

Markscheme

correct substitution into the formula for volume ***A1***

eg $36 = y \times x \times x$

valid approach to eliminate y (may be seen in formula/substitution) ***M1***

eg $y = \frac{36}{x^2}$, $xy = \frac{36}{x}$

correct expression for surface area ***A1***

eg $xy + xy + xy + x^2 + x^2$, $\text{area} = 3xy + 2x^2$

correct expression in terms of x only ***A1***

eg $3x\left(\frac{36}{x^2}\right) + 2x^2$, $x^2 + x^2 + \frac{36}{x} + \frac{36}{x} + \frac{36}{x}$, $2x^2 + 3\left(\frac{36}{x}\right)$

$A(x) = \frac{108}{x} + 2x^2$ ***AG*** ***N0***

[4 marks]

19b. Find $A'(x)$. *[2 marks]*

Markscheme

$A'(x) = -\frac{108}{x^2} + 4x$, $4x - 108x^{-2}$ ***A1A1*** ***N2***

Note: Award ***A1*** for each term.

[2 marks]

19c. Given that the outside surface area is a minimum, find the height of the container. *[5 marks]*

Markscheme

recognizing that minimum is when $A'(x) = 0$ ***(M1)***

correct equation ***(A1)***

eg $-\frac{108}{x^2} + 4x = 0$, $4x = \frac{108}{x^2}$

correct simplification ***(A1)***

eg $-108 + 4x^3 = 0$, $4x^3 = 108$

correct working ***(A1)***

eg $x^3 = 27$

$\text{height} = 3\ (\text{m})\ (\text{accept } x = 3)$ ***A1*** ***N2***

[5 marks]

19d. Fred paints the outside of the container. A tin of paint covers a surface area of $10\ \mathrm{m}^2$ *[5 marks]* and costs $20. Find the total cost of the tins needed to paint the container.

Markscheme

attempt to find area using **their** height ***(M1)***

eg $\frac{108}{3} + 2(3)^2,\ 9 + 9 + 12 + 12 + 12$

minimum surface area $= 54\ \mathrm{m}^2$ (may be seen in part (c)) ***A1***

attempt to find the number of tins ***(M1)***

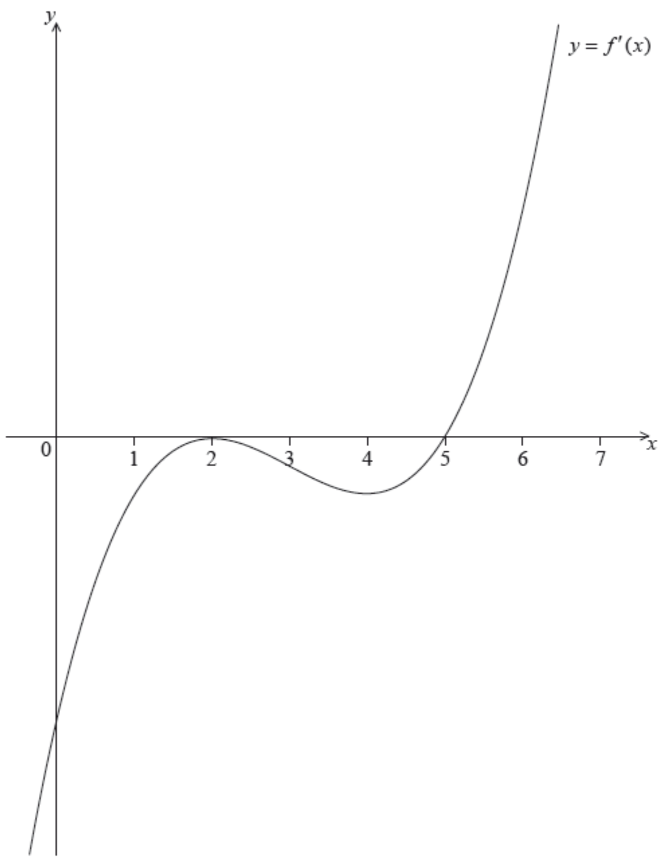
eg $\frac{54}{10},\ 5.4$

6 (tins) ***(A1)***

$120 ***A1*** ***N3***

[5 marks]

Let $y = f(x)$, for $-0.5 \le x \le 6.5$. The following diagram shows the graph of f', the derivative of f.



The graph of f' has a local maximum when $x = 2$, a local minimum when $x = 4$, and it crosses the
x-axis at the point $(5,\ 0)$.

20a. Explain why the graph of f has a local minimum when $x = 5$. *[2 marks]*

Markscheme

METHOD 1

$f'(5) = 0$ ***(A1)***

valid reasoning including reference to the graph of f' ***R1***

eg f' changes sign from negative to positive at $x = 5$, labelled sign chart for f'

so f has a local minimum at $x = 5$ ***AG*** ***N0***

Note: It must be clear that any description is referring to the graph of f', simply giving the conditions for a minimum without relating them to f' does not gain the ***R1***.

METHOD 2

$f'(5) = 0$ ***A1***

valid reasoning referring to second derivative ***R1***

eg $f''(5) > 0$

so f has a local minimum at $x = 5$ ***AG*** ***N0***

[2 marks]

20b. Find the set of values of x for which the graph of f is concave down. *[2 marks]*

Markscheme

attempt to find relevant interval ***(M1)***

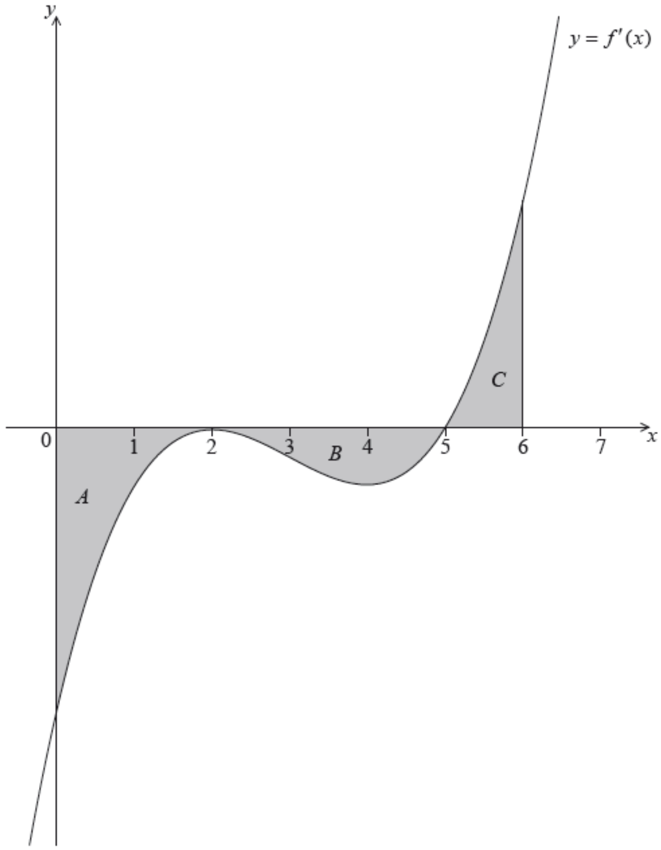
eg f' is decreasing, gradient of f' is negative, $f'' < 0$

$2 < x < 4$ (accept "between 2 and 4") ***A1*** ***N2***

Notes: If no other working shown, award ***M1A0*** for incorrect inequalities such as $2 \leq x \leq 4$, or "from 2 to 4"

[2 marks]

20c. The following diagram shows the shaded regions A, B and C. *[5 marks]*



The regions are enclosed by the graph of f', the x-axis, the y-axis, and the line $x = 6$.

The area of region A is 12, the area of region B is 6.75 and the area of region C is 6.75.

Given that $f(0) = 14$, find $f(6)$.

Markscheme

METHOD 1 (one integral)

correct application of Fundamental Theorem of Calculus ***(A1)***

eg $\int_0^6 f'(x)\mathrm{d}x = f(6) - f(0),\ f(6) = 14 + \int_0^6 f'(x)\mathrm{d}x$

attempt to link definite integral with areas ***(M1)***

eg $\int_0^6 f'(x)\mathrm{d}x = -12 - 6.75 + 6.75,\ \int_0^6 f'(x)\mathrm{d}x = \text{Area } A + \text{Area } B + \text{ Area } C$

correct value for $\int_0^6 f'(x)\mathrm{d}x$ ***(A1)***

eg $\int_0^6 f'(x)\mathrm{d}x = -12$

correct working ***A1***

eg $f(6) - 14 = -12,\ f(6) = -12 + f(0)$

$f(6) = 2$ ***A1 N3***

METHOD 2 (more than one integral)

correct application of Fundamental Theorem of Calculus ***(A1)***

eg $\int_0^2 f'(x)\mathrm{d}x = f(2) - f(0),\ f(2) = 14 + \int_0^2 f'(x)$

attempt to link definite integrals with areas ***(M1)***

eg $\int_0^2 f'(x)\mathrm{d}x = 12,\ \int_2^5 f'(x)\mathrm{d}x = -6.75,\ \int_0^6 f'(x) = 0$

correct values for integrals ***(A1)***

eg $\int_0^2 f'(x)\mathrm{d}x = -12,\ \int_5^2 f'(x)\mathrm{d}x = 6.75,\ f(6) - f(2) = 0$

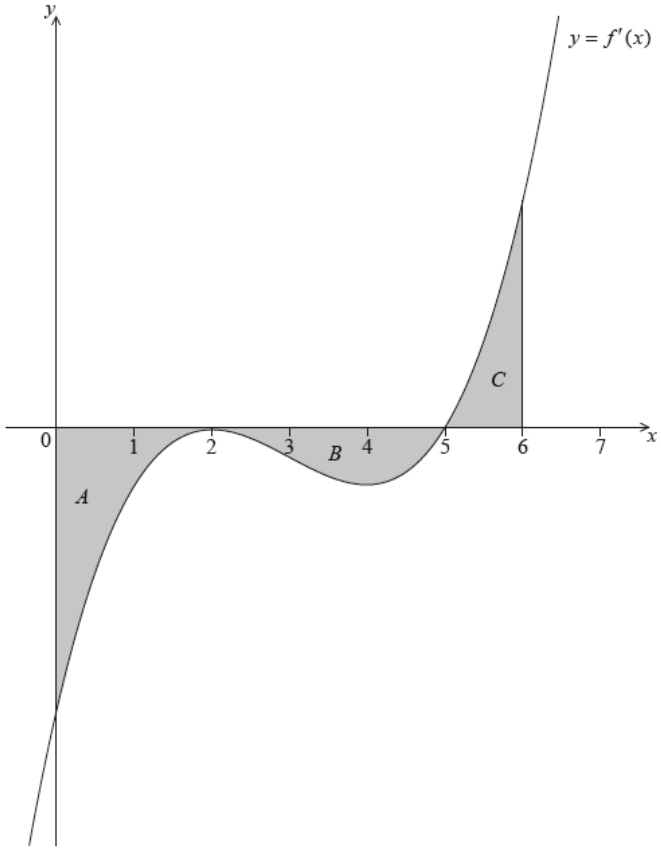
one correct intermediate value ***A1***

eg $f(2) = 2,\ f(5) = -4.75$

$f(6) = 2$ ***A1 N3***

[5 marks]

20d. The following diagram shows the shaded regions A, B and C. *[6 marks]*



The regions are enclosed by the graph of f', the *x*-axis, the *y*-axis, and the line $x = 6$.

The area of region A is 12, the area of region B is 6.75 and the area of region C is 6.75.

Let $g(x) = (f(x))^2$. Given that $f'(6) = 16$, find the equation of the tangent to the graph of g at the point where $x = 6$.

Markscheme

correct calculation of $g(6)$ (seen anywhere) ***A1***

eg $2^2,\ g(6) = 4$

choosing chain rule or product rule ***(M1)***

eg $g'\left(f(x)\right) f'(x),\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x},\ f(x)f'(x) + f'(x)f(x)$

correct derivative ***(A1)***

eg $g'(x) = 2f(x)f'(x),\ f(x)f'(x) + f'(x)f(x)$

correct calculation of $g'(6)$ (seen anywhere) ***A1***

eg $2(2)(16),\ g'(6) = 64$

attempt to substitute **their** values of $g'(6)$ and $g(6)$ (in any order) into equation of a line ***(M1)***

eg $2^2 = (2 \times 2 \times 16)6 + b,\ y - 6 = 64(x - 4)$

correct equation in any form ***A1*** ***N2***

eg $y - 4 = 64(x - 6),\ y = 64x - 380$

[6 marks]

[Total 15 marks]

A function f has its derivative given by $f'(x) = 3x^2 - 2kx - 9$, where k is a constant.

21a. Find $f''(x)$. *[2 marks]*

Markscheme

$f''(x) = 6x - 2k$ ***A1A1*** ***N2***

[2 marks]

21b. The graph of f has a point of inflexion when $x = 1$. *[3 marks]*

Show that $k = 3$.

Markscheme

substituting $x = 1$ into f'' ***(M1)***

eg $f''(1),\ 6(1) - 2k$

recognizing $f''(x) = 0$ (seen anywhere) ***M1***

correct equation ***A1***

eg $6 - 2k = 0$

$k = 3$ ***AG*** ***N0***

[3 marks]

21c. Find $f'(-2)$. *[2 marks]*

Markscheme

correct substitution into $f'(x)$ ***(A1)***

eg $3(-2)^2 - 6(-2) - 9$

$f'(-2) = 15$ ***A1*** ***N2***

[2 marks]

21d. Find the equation of the tangent to the curve of f at $(-2,\ 1)$, giving your answer in the form $y = ax + b$. *[4 marks]*

Markscheme

recognizing gradient value (may be seen in equation) ***M1***

eg $a = 15,\ y = 15x + b$

attempt to substitute $(-2,\ 1)$ into equation of a straight line ***M1***

eg $1 = 15(-2) + b,\ (y - 1) = m(x + 2),\ (y + 2) = 15(x - 1)$

correct working ***(A1)***

eg $31 = b,\ y = 15x + 30 + 1$

$y = 15x + 31$ ***A1*** ***N2***

[4 marks]

21e. Given that $f'(-1) = 0$, explain why the graph of f has a local maximum when $x = -1$. *[3 marks]*

Markscheme

METHOD 1 (2^{nd} derivative)

recognizing $f'' < 0$ (seen anywhere) ***R1***

substituting $x = -1$ into f'' ***(M1)***

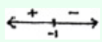
eg $f''(-1),\ 6(-1) - 6$

$f''(-1) = -12$ ***A1***

therefore the graph of f has a local maximum when $x = -1$ ***AG*** ***N0***

METHOD 2 (1^{st} derivative)

recognizing change of sign of $f'(x)$ (seen anywhere) ***R1***

eg *sign chart* 

correct value of f' for $-1 < x < 3$ ***A1***

eg $f'(0) = -9$

correct value of f' for x value to the left of -1 ***A1***

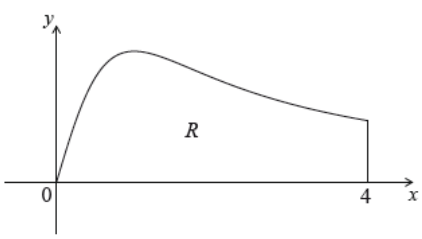
eg $f'(-2) = 15$

therefore the graph of f has a local maximum when $x = -1$ ***AG*** ***N0***

[3 marks]

Total [14 marks]

22. The following diagram shows the graph of $f(x) = \frac{x}{x^2+1}$, for $0 \leq x \leq 4$, and the line $x = 4$. *[6 marks]*



Let R be the region enclosed by the graph of f, the x-axis and the line $x = 4$.

Find the area of R.

Markscheme

substitution of limits or function ***(A1)***

eg $A = \int_0^4 f(x),\ \int \frac{x}{x^2+1} \mathrm{d}x$

correct integration by substitution/inspection ***A2***

$\frac{1}{2} \ln(x^2+1)$

substituting limits into **their** integrated function and subtracting (in any order) ***(M1)***

eg $\frac{1}{2}\left(\ln(4^2+1) - \ln(0^2+1)\right)$

correct working ***A1***

eg $\frac{1}{2}\left(\ln(4^2+1) - \ln(0^2+1)\right),\ \frac{1}{2}(\ln(17) - \ln(1)),\ \frac{1}{2}\ln 17 - 0$

$A = \frac{1}{2}\ln(17)$ ***A1*** ***N3***

Note: Exception to ***FT*** rule. Allow full ***FT*** on incorrect integration involving a $\ln$ function.

[6 marks]

Let
$f(x) = px^3 + px^2 + qx$.

23a. Find $f'(x)$. *[2 marks]*

Markscheme

$f'(x) = 3px^2 + 2px + q$ ***A2*** ***N2***

Note: Award ***A1*** if only 1 error.

[2 marks]

23b. Given that $f'(x) \geqslant 0$, show that $p^2 \leqslant 3pq$. *[5 marks]*

Markscheme

evidence of discriminant (must be seen explicitly, not in quadratic formula) ***(M1)***

eg $b^2 - 4ac$

correct substitution into discriminant (may be seen in inequality) ***A1***

eg $(2p)^2 - 4 \times 3p \times q,\ 4p^2 - 12pq$

$f'(x) \geqslant 0$ then f' has two equal roots or no roots ***(R1)***

recognizing discriminant less or equal than zero ***R1***

eg $\Delta \leqslant 0,\ 4p^2 - 12pq \leqslant 0$

correct working that clearly leads to the required answer ***A1***

eg $p^2 - 3pq \leqslant 0,\ 4p^2 \leqslant 12pq$

$p^2 \leqslant 3pq$ ***AG*** ***N0***

[5 marks]

Let
$f(x) = \frac{2x}{x^2+5}$.

24a. Use the quotient rule to show that $f'(x) = \frac{10-2x^2}{(x^2+5)^2}$. *[4 marks]*

Markscheme

derivative of $2x$ is 2 (must be seen in quotient rule) ***(A1)***

derivative of $x^2 + 5$ is $2x$ (must be seen in quotient rule) ***(A1)***

correct substitution into quotient rule ***A1***

eg $\frac{(x^2+5)(2)-(2x)(2x)}{(x^2+5)^2},\ \frac{2(x^2+5)-4x^2}{(x^2+5)^2}$

correct working which clearly leads to given answer ***A1***

eg $\frac{2x^2+10-4x^2}{(x^2+5)^2},\ \frac{2x^2+10-4x^2}{x^4+10x^2+25}$

$f'(x) = \frac{10-2x^2}{(x^2+5)^2}$ ***AG*** ***N0***

[4 marks]

24b. Find $\int \frac{2x}{x^2+5}\mathrm{d}x$. *[4 marks]*

Markscheme

valid approach using substitution or inspection ***(M1)***

eg $u = x^2 + 5,\ \mathrm{d}u = 2x\mathrm{d}x,\ \frac{1}{2}\ln(x^2+5)$

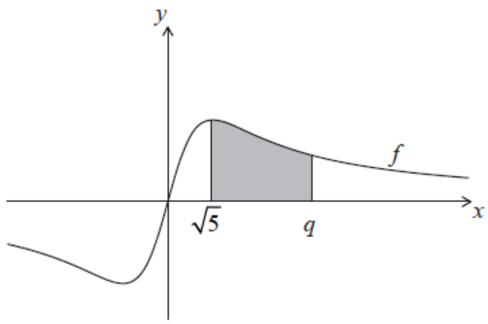
$\int \frac{2x}{x^2+5}\mathrm{d}x = \int \frac{1}{u}\mathrm{d}u$ ***(A1)***

$\int \frac{1}{u}\mathrm{d}u = \ln u + c$ ***(A1)***

$\ln(x^2+5) + c$ ***A1 N4***

[4 marks]

24c. The following diagram shows part of the graph of f. *[7 marks]*



The shaded region is enclosed by the graph of f, the x-axis, and the lines $x = \sqrt{5}$ and $x = q$. This region has an area of $\ln 7$. Find the value of q.

Markscheme

correct expression for area ***(A1)***

eg $\left[\ln\left(x^2+5\right)\right]_{\sqrt{5}}^{q},\ \int\limits_{\sqrt{5}}^{q}\frac{2x}{x^2+5}\mathrm{d}x$

substituting limits into **their** integrated function and subtracting (in either order) ***(M1)***

eg $\ln(q^2+5)-\ln\left(\sqrt{5}^2+5\right)$

correct working ***(A1)***

eg $\ln\left(q^2+5\right)-\ln 10,\ \ln\frac{q^2+5}{10}$

equating **their** expression to $\ln 7$ (seen anywhere) ***(M1)***

eg $\ln\left(q^2+5\right)-\ln 10=\ln 7,\ \ln\frac{q^2+5}{10}=\ln 7,\ \ln(q^2+5)=\ln 7+\ln 10$

correct equation without logs ***(A1)***

eg $\frac{q^2+5}{10}=7,\ q^2+5=70$

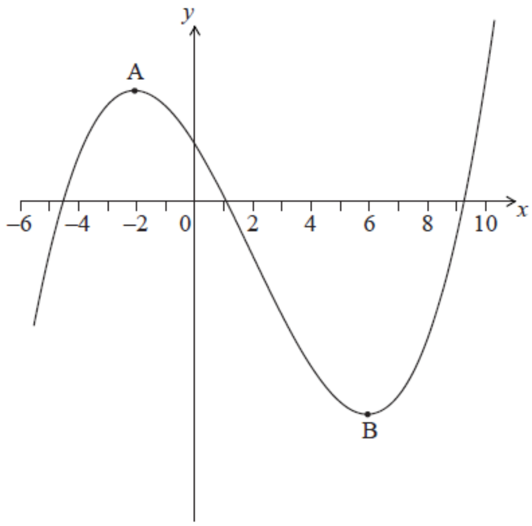
$q^2=65$ ***(A1)***

$q=\sqrt{65}$ ***A1 N3***

Note: Award ***A0*** for $q=\pm\sqrt{65}$.

[7 marks]

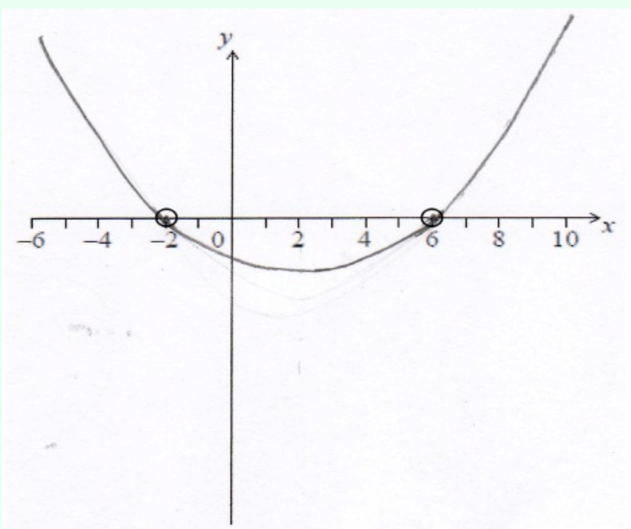
The following diagram shows part of the graph of
$y = f(x)$.



The graph has a local maximum at
A, where
$x = -2$, and a local minimum at
B, where
$x = 6$.

25a. On the following axes, sketch the graph of $y = f'(x)$. *[4 marks]*

Markscheme



A1A1A1A1 ***N4***

Note: Award ***A1*** for *x*-intercept in circle at -2, ***A1*** for *x*-intercept in circle at 6.

Award ***A1*** for approximately correct shape.

Only if this ***A1*** is awarded, award ***A1*** for a negative *y*-intercept.

[4 marks]

25b. Write down the following in order from least to greatest: $f(0),\ f'(6),\ f''(-2)$. *[2 marks]*

Markscheme

$f''(-2),\ f'(6),\ f(0)$ ***A2*** ***N2***

[2 marks]

Consider the functions
$f(x)$,
$g(x)$ and
$h(x)$. The following table gives some valuesassociated with these functions.

x	2	3
$f(x)$	2	3
$g(x)$	–14	–18
$f'(x)$	1	1
$g'(x)$	–5	–3
$h''(x)$	–6	0

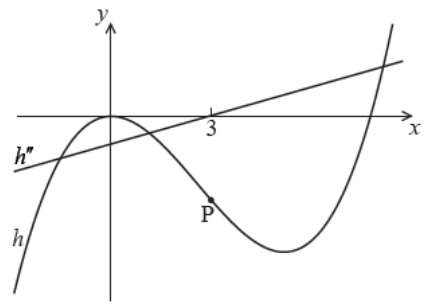
26a. Write down the value of $g(3)$, of $f'(3)$, and of $h''(2)$. *[3 marks]*

Markscheme

$g(3) = -18$, $f'(3) = 1$, $h''(2) = -6$ ***A1A1A1*** ***N3***

[3 marks]

The following diagram shows parts of the graphs of
h and
h'' .



There is a point of inflexion on the graph of
h at P, when
$x = 3$.

26b. Explain why P is a point of inflexion. *[2 marks]*

Markscheme

$h''(3) = 0$ ***(A1)***

valid reasoning ***R1***

eg h'' changes sign at $x = 3$, change in concavity of h at $x = 3$

so P is a point of inflexion ***AG N0***

[2 marks]

Given that
$h(x) = f(x) \times g(x)$,

26c. find the y-coordinate of P. *[2 marks]*

Markscheme

writing $h(3)$ as a product of $f(3)$ and $g(3)$ ***A1***

eg $f(3) \times g(3)$, $3 \times (-18)$

$h(3) = -54$ ***A1 N1***

[2 marks]

26d. find the equation of the normal to the graph of h at P. *[7 marks]*

Markscheme

recognizing need to find derivative of h ***(R1)***

eg h' , $h'(3)$

attempt to use the product rule (do **not** accept $h' = f' \times g'$) ***(M1)***

eg $h' = fg' + gf'$, $h'(3) = f(3) \times g'(3) + g(3) \times f'(3)$

correct substitution ***(A1)***

eg $h'(3) = 3(-3) + (-18) \times 1$

$h'(3) = -27$ ***A1***

attempt to find the gradient of the normal ***(M1)***

eg $-\frac{1}{m}$, $-\frac{1}{27}x$

attempt to substitute **their** coordinates and **their** normal gradient into the equation of a line ***(M1)***

eg $-54 = \frac{1}{27}(3) + b$, $0 = \frac{1}{27}(3) + b$, $y + 54 = 27(x - 3)$, $y - 54 = \frac{1}{27}(x + 3)$

correct equation in any form ***A1*** ***N4***

eg $y + 54 = \frac{1}{27}(x - 3)$, $y = \frac{1}{27}x - 54\frac{1}{9}$

[7 marks]

27. A rocket moving in a straight line has velocity v km s^{-1} and displacement s km at time t seconds. The velocity v is given by $v(t) = 6\mathrm{e}^{2t} + t$. When $t = 0$, $s = 10$. *[7 marks]*

Find an expression for the displacement of the rocket in terms of t .

Markscheme

evidence of anti-differentiation ***(M1)***

eg $\int (6\mathrm{e}^{2t} + t)$

$s = 3\mathrm{e}^{2t} + \frac{t^2}{2} + C$ ***A2A1***

Note: Award ***A2*** for $3\mathrm{e}^{2t}$, ***A1*** for $\frac{t^2}{2}$.

attempt to substitute $(0, 10)$ into **their** integrated expression (even if C is missing) ***(M1)***

correct working ***(A1)***

eg $10 = 3 + C$, $C = 7$

$s = 3\mathrm{e}^{2t} + \frac{t^2}{2} + 7$ ***A1*** ***N6***

Note: Exception to the ***FT*** rule. If working shown, allow full ***FT*** on incorrect integration which must involve a power of e.

[7 marks]

Let
$f(x) = \sin x + \frac{1}{2}x^2 - 2x$, for
$0 \leq x \leq \pi$.

28a. Find $f'(x)$. *[3 marks]*

Markscheme

$f'(x) = \cos x + x - 2$ ***A1A1A1*** ***N3***

Note: Award ***A1*** for each term.

[3 marks]

Let
g be a quadratic function such that
$g(0) = 5$. The line
$x = 2$ is the axis of symmetry of the graph of
g .

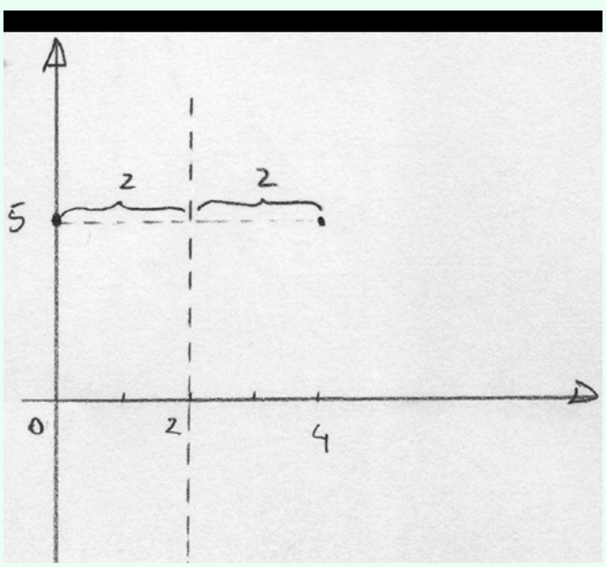
28b. Find $g(4)$. *[3 marks]*

Markscheme

recognizing $g(0) = 5$ gives the point $(0, 5)$ ***(R1)***

recognize symmetry ***(M1)***

eg vertex, sketch



$g(4) = 5$ ***A1*** ***N3***

[3 marks]

The function
g can be expressed in the form
$g(x) = a(x - h)^2 + 3$.

28c. (i) Write down the value of h . *[4 marks]*

(ii) Find the value of a .

Markscheme

(i) $h = 2$ ***A1 N1***

(ii) substituting into $g(x) = a(x - 2)^2 + 3$ (not the vertex) ***(M1)***

eg $5 = a(0 - 2)^2 + 3$, $5 = a(4 - 2)^2 + 3$

working towards solution ***(A1)***

eg $5 = 4a + 3$, $4a = 2$

$a = \frac{1}{2}$ ***A1 N2***

[4 marks]

28d. Find the value of x for which the tangent to the graph of f is parallel to the tangent to the graph of g . *[6 marks]*

Markscheme

$g(x) = \frac{1}{2}(x - 2)^2 + 3 = \frac{1}{2}x^2 - 2x + 5$

correct derivative of g ***A1A1***

eg $2 \times \frac{1}{2}(x - 2)$, $x - 2$

evidence of equating both derivatives ***(M1)***

eg $f' = g'$

correct equation ***(A1)***

eg $\cos x + x - 2 = x - 2$

working towards a solution ***(A1)***

eg $\cos x = 0$, combining like terms

$x = \frac{\pi}{2}$ ***A1 N0***

Note: Do not award final ***A1*** if additional values are given.

[6 marks]